

**$L^p$ - $L^2$  FOURIER RESTRICTION FOR HYPERSURFACES IN  $\mathbb{R}^3$  :**  
**PART II**  
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ISROIL A. IKROMOV AND DETLEF MÜLLER

ABSTRACT. This is the second in a series of two articles, in which we prove a sharp  $L^p$ - $L^2$  Fourier restriction theorem for a large class of smooth, finite type hypersurfaces in  $\mathbb{R}^3$ , which includes in particular all real-analytic hypersurfaces.

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## 1. INTRODUCTION

This is the second part of a pair of articles whose main goal is to prove our main result, Theorem 1.7, in [21] on  $L^p$ - $L^2$  Fourier restriction estimates for smooth hypersurfaces of finite type in  $\mathbb{R}^3$ . For the relevant statements, definitions and bibliographical references we therefore refer the reader to the introduction to that article. Under the assumption that our hypersurface  $S$  is given as the graph of a smooth function  $\phi$  defined near the origin and satisfying the conditions  $\phi(0,0) = 0$  and  $\nabla\phi(0,0) = 0$ , we had covered in [21] all situations with the exception of the cases where the so-called linear height  $h_{\text{lin}}(\phi)$  satisfies  $2 \leq h_{\text{lin}}(\phi) < 5$ . For this case, substantially more refined methods than the ones used in [21] are needed, since the use of Drury's restriction estimate for non-degenerate curves turns out to be insufficient. In fact, the method that we shall develop in this second part will work whenever  $h_{\text{lin}}(\phi) \geq 2$ .

Throughout this article, we shall make the following general

**Assumption 1.1.** *There is no linear coordinate system which is adapted to  $\phi$ .*

Moreover, we may and shall assume that we are in linearly adapted coordinates, so that  $d = h_{\text{lin}}(\phi) \geq 2$ . Recall also from [21] that this assumption implies that the principal face  $\pi(\phi)$  of the Newton polyhedron of  $\phi$  is a compact edge which is intersected by the bi-sectrix

$$\Delta := \{(t_1, t_2) \in \mathbb{R}^2 : t_1 = t_2\},$$

in an interior point, given by  $(d, d)$ , and that  $\pi(\phi)$  is contained in the *principal line*

$$L := \{(t_1, t_2) \in \mathbb{R}^2 : \kappa_1 t_1 + \kappa_2 t_2 = 1\},$$

and thus determines a weight  $\kappa := (\kappa_1, \kappa_2)$ , so that also  $m = \kappa_2/\kappa_1 \geq 2$ .

**Conventions:** As in [21], we shall use the “variable constant” notation in this article, i.e., many constants appearing in the paper, often denoted by  $C$ , will typically have different values at different lines. Moreover, we shall use symbols such as  $\sim$ ,  $\lesssim$  or  $\ll$  in order to avoid writing down constants. By  $A \sim B$  we mean that there are constants  $0 < C_1 \leq C_2$  such that  $C_1 A \leq B \leq C_2 A$ , and these constants will not

depend on the relevant parameters arising in the context in which the quantities  $A$  and  $B$  appear. Similarly, by  $A \lesssim B$  we mean that there is a (possibly large) constant  $C_1 > 0$  such that  $A \leq C_1 B$ , and by  $A \ll B$  we mean that there is a sufficiently small constant  $c_1 > 0$  such that  $A \leq c_1 B$ , and again these constants do not depend on the relevant parameters.

By  $\chi_0$  and  $\chi_1$  we shall always denote smooth cut-off functions with compact support on  $\mathbb{R}^n$ , where  $\chi_0$  will be supported in a neighborhood of the origin, whereas  $\chi_1 = \chi_1(x)$  will be support away from the origin in each of its coordinates  $x_j$ , i.e.,  $|x_j| \sim 1$  for every  $j = 1, \dots, n$ . These cut-off functions may also vary from line to line, and may in some instances, where several of such functions of different variables appear within the same formula, even designate different functions.

Also, if we speak of the *slope* of a line such as a supporting line to a Newton polyhedron, then we shall actually mean the modulus of the slope.

## 2. THE CASE WHEN $h_{\text{lin}}(\phi) \geq 2$ : REMINDER OF THE OPEN CASES

Recall from [21], Section 9, the following two **Cases**:

- (a) The principal face  $\pi(\phi^a)$  of the Newton polyhedron  $\mathcal{N}(\phi^a)$  of  $\phi^a$  is a compact edge, which lies on a line  $L^a$ , which we call the *principal line* of  $\mathcal{N}(\phi^a)$
- (b)  $\pi(\phi^a)$  is the vertex  $(h, h)$ .

What had remained open in [21] was the study of the piece of the surface  $S$  corresponding to the domain  $D_{\text{pr}}$  containing the principal root jet  $\psi$ , in the cases (a) and (b), i.e.,

$$(2.1) \quad D_{\text{pr}} := \begin{cases} \{(x_1, x_2) : 0 < x_1 < \varepsilon, |x_2 - \psi(x_1)| \leq Nx_1^a\} & \text{in Case (a),} \\ \{(x_1, x_2) : 0 < x_1 < \varepsilon, |x_2 - \psi(x_1)| \leq \varepsilon x_1^a\}, & \text{in Case (b),} \end{cases}$$

when  $2 \leq h_{\text{lin}}(\phi) < 5$ . Indeed, we shall here develop an approach which will work whenever  $h_{\text{lin}}(\phi) \geq 2$ . Our goal will thus be to prove the following extension of Proposition 12.1 in [21] to the case where  $h_{\text{lin}}(\phi) \geq 2$ :

**Proposition 2.1.** *Assume that  $h_{\text{lin}}(\phi) \geq 2$ , and that we are in Case (a) or (b). When  $\varepsilon > 0$  is sufficiently small, and  $N$  is sufficiently large in Case (a), then*

$$\left( \int_{D_{\text{pr}}} |\widehat{f}|^2 d\mu \right)^{1/2} \leq C_p \|f\|_{L^p(\mathbb{R}^3)}, \quad f \in \mathcal{S}(\mathbb{R}^3),$$

whenever  $p' \geq p'_c$ .

In order to prove this proposition, we follow the domain decomposition algorithm for the domain  $D_{\text{pr}}$  developed in Section 12 of [21]. In Case (a), that algorithm led to a finite family of subdomains  $E_{(l)}$  (so-called transition domains) and domains  $D'_{(l)}$ ,  $l \geq 1$ , of the form

$$D'_{(l)} := \{(x_1, x_2) : 0 < x_1 < \delta, |x_2 - \psi^{(l+1)}(x_1)| \leq \varepsilon x_1^{a_{(l)}}\},$$

where the functions  $\psi^{(l)}$  are of the form

$$\psi^{(l)}(x_1) = \psi(x_1) + \sum_{j=1}^{l-1} c_{j-1} x_1^{a_{(j)}},$$

with real coefficients  $c_j$ , and where the exponents  $a_{(j)}$  form a strictly increasing sequence

$$a = a_{(1)} < a_{(2)} < \cdots < a_{(l)} < a_{(l+1)} < \cdots$$

of rational numbers. Moreover, in the modified adapted coordinates given by

$$y_1 := x_1, \quad y_2 := x_2 - \psi^{(l+1)}(x_1),$$

the function  $\phi$  is given by

$$\phi^{(l+1)}(y_1, y_2) := \phi(y_1, y_2 + \psi^{(l+1)}(y_1)).$$

Notice that we can define these notions also for  $l = 0$ , and then have  $\psi^{(0)} = \psi$  and  $\phi^{(0)} = \phi^a$ .

Moreover, the domain  $D'_{(l)}$  is associated to an “edge”  $\gamma'_{(l)} = [(A'_{(l-1)}, B'_{(l-1)}), (A'_{(l)}, B'_{(l)})]$  (which is indeed an edge, or can degenerate to a single point) of the Newton polyhedron of  $\phi^{(l+1)}$  in the following way:

The edge with index  $l$  will lie on a line

$$L_{(l)} := \{(t_1, t_2) \in \mathbb{R}^2 : \kappa_1^{(l)} t_1 + \kappa_2^{(l)} t_2 = 1\},$$

of slope  $1/a_{(l)}$  (here, we shall always mean the modulus of the slope), where  $a_{(l)} = \kappa_2^{(l)}/\kappa_1^{(l)}$ . Introduce corresponding “ $\kappa^{(l)}$ -dilations”  $\delta_r = \delta_r^{\kappa^{(l)}}$  by putting  $\delta_r(y_1, y_2) = (r^{\kappa_1^{(l)}} y_1, r^{\kappa_2^{(l)}} y_2)$ ,  $r > 0$ . Then the domain

$$D'^a_{(l)} := \{(y_1, y_2) : 0 < y_1 < \varepsilon, |y_2| \leq \varepsilon y_1^{a_{(l)}}\},$$

which represents the domain  $D'_{(l)}$  in the coordinates  $(y_1, y_2)$ , is invariant under these dilations, and the Newton diagram of the  $\kappa^{(l)}$ -principal part  $\phi_{\kappa^{(l)}}^{(l+1)}$  of  $\phi^{(l+1)}$  agrees with the edge  $\gamma'_{(l)}$ .

Recall also that the first edge  $\gamma'_{(1)}$  agrees with the principal face  $\pi(\phi^{(2)})$  of  $\phi^{(2)}$  and lies on the principal line  $L^a$  of the Newton polyhedron of  $\phi^a$ , and it intersects the bi-sectrix  $\Delta$ , whereas for  $l \geq 2$  the edge  $\gamma'_{(l)}$  will lie in the closed half-space below the bi-sectrix.

Moreover, the Newton polyhedra of  $\phi^a$  and of  $\phi^{(l)}$  do agree in the closed half-space above the bi-sectrix.

Now, in Section 8 of [21], setting  $v = (1, 0)$ , we had distinguished between the cases where  $\partial_2 \phi_{\kappa^{(l)}}^{(l+1)}(v) \neq 0$  (Case 1),  $\partial_2 \phi_{\kappa^{(l)}}^{(l+1)}(v) = 0$  and  $\partial_1 \phi_{\kappa^{(l)}}^{(l+1)}(v) \neq 0$  (Case 2), and the case where  $\nabla \phi_{\kappa^{(l)}}^{(l+1)}(v) = 0$  (Case 3), and studied restriction estimates for the pieces of the surface  $S$  corresponding to the domain  $D'_{(l)}$ .

Only in Case 2 we had made use of the assumption  $h_{\text{lin}}(\phi) \geq 5$ , so we can concentrate in the sequel on Case 2.

Notice also that our decomposition algorithm worked as well in Case (b), only that we had to skip the first step of the algorithm. We shall therefore first study the domain  $D'_{(1)}$  in Case (a), and in the last section describe the minor modifications needed to treat also the domains  $D'_{(l)}$  for  $l \geq 2$ , which will then also cover Case (b) at the same time.

We can localize to the domain  $D'_{(l)}$  by means of a cut-off function

$$\rho_{(l)}(x_1, x_2) := \chi_0\left(\frac{x_2 - \psi^{(l+1)}(x_1)}{\varepsilon x_1^{a_{(l)}}}\right),$$

where  $\chi_0 \in \mathcal{D}(\mathbb{R})$ . Let us again fix a suitable smooth cut-off function  $\chi \geq 0$  on  $\mathbb{R}^2$  supported in an annulus  $\mathcal{A} := \{x \in \mathbb{R}^2 : 1/2 \leq |y| \leq R\}$  such that the functions  $\chi_k^a := \chi \circ \delta_{2^k}$  form a partition of unity. Here,  $\delta_r = \delta_r^{\kappa^{(l)}}$  denote the dilations associated to the weight  $\kappa^{(l)}$ . In the original coordinates  $x$ , these correspond to the functions  $\chi_k(x) := \chi_k^a(x_1, x_2 - \psi^{(l+1)}(x_1))$ . We then decompose the measure  $\mu^{\rho_{(c_0)}}$  dyadically as

$$(2.2) \quad \mu^{\rho_{(l)}} = \sum_{k \geq k_0} \mu_k,$$

where

$$\mu_k := \mu_k^{(l)} := \mu^{\chi_k \rho_{(l)}}.$$

Notice that by choosing the support of  $\eta$  sufficiently small, we can choose  $k_0 \in \mathbb{N}$  as large as we need. It is also important to observe that this decomposition can essentially be achieved by means of a dyadic decomposition with respect to the variable  $x_1$ , which again allows to apply Littlewood-Paley theory (see [21]).

Moreover, changing to modified adapted coordinates in the integral defining  $\mu_k$  and scaling by  $\delta_{2^{-k}}$  we find that

$$(2.3) \quad \langle \mu_k, f \rangle = 2^{-k|\kappa^{(l)}|} \int f(2^{-\kappa_1^{(l)}k}x_1, 2^{-\kappa_2^{(l)}k}x_2 + 2^{-m\kappa_1^{(l)}k}x_1^m \omega(2^{-\kappa_1^{(l)}k}x_1), 2^{-k}\phi_k(x)) \eta(x) dx,$$

where  $\omega = \omega^{(l)}$  is given by

$$\psi^{(l+1)}(x_1) = x_1^m \omega(x_1),$$

so that  $\omega(0) \neq 0$ ,  $\eta = \eta_k^{(l)}$  is a smooth function supported where  $x_1 \sim 1$ ,  $|x_2| < \varepsilon$  (for some small  $\varepsilon > 0$ ), whose derivatives are uniformly bounded  $k$ , and where

$$(2.4) \quad \phi_k(x) = \phi_k^{(l+1)}(x) := 2^k \phi^{(l+1)}(\delta_{2^{-k}}x) = \phi_{\kappa^{(l)}}^{(l+1)}(x) + \text{error terms of order } O(2^{-\delta k})$$

with respect to the  $C^\infty$  topology (and  $\delta > 0$ ).

In order to prove Proposition 2.1, we then still need to prove

**Proposition 2.2.** *Assume that  $h_{\text{lin}} \geq 2$ , that we are in Case 2, i.e.,  $\partial_2 \phi_{\kappa^{(l)}}^{(l+1)}(1, 0) = 0$  and  $\partial_1 \phi_{\kappa^{(l)}}^{(l+1)}(1, 0) \neq 0$ , and recall that  $p'_c = 2h^r + 2$ . When  $\varepsilon > 0$  is sufficiently small and  $k_0 \in \mathbb{N}$  is sufficiently large, then for every  $l \geq 1$ ,*

$$(2.5) \quad \left( \int |\widehat{f}|^2 d\mu_k \right)^{1/2} \leq C_{p_c} \|f\|_{L^p(\mathbb{R}^3)}, \quad f \in \mathcal{S}(\mathbb{R}^3), \quad k \geq k_0,$$

where the constant  $C_p$  is independent of  $k$ .

### 3. RESTRICTION ESTIMATES FOR THE DOMAIN $D'_{(1)}$

Let us assume that we are in Case (a), where the principal face  $\pi(\phi^a)$  is a compact edge. In the enumeration of edges  $\gamma_l$  of the Newton polyhedron associated to  $\phi^a$  in Section 7 of [21], this edge corresponds to the index  $l = l_{\text{pr}}$ , i.e.,

$$(3.1) \quad \pi(\phi^a) = \gamma_{l_{\text{pr}}}.$$

The weight  $\kappa^{(1)}$  is here the principal weight  $\kappa^{l_{\text{pr}}}$  from [21], and the line  $L_{(1)}$  is the principal line  $L^a = L_{l_{\text{pr}}}$  of the Newton polyhedron of  $\phi^a$ . We then put

$$\tilde{\kappa} := \kappa^{(1)}, \quad \text{so that} \quad a = \frac{\tilde{\kappa}_2}{\tilde{\kappa}_1}, \quad \phi_{\tilde{\kappa}}^a = \phi_{\text{pr}}^a.$$

In particular,  $h_{l_{\text{pr}}} + 1$  is the second coordinate of the point of intersection of the line

$$\Delta^{(m)} := \{(t, t + m + 1) : t \in \mathbb{R}\}$$

with the line  $L^a$ , and according to [21], display (1.11), is given by

$$(3.2) \quad h_{l_{\text{pr}}} + 1 = \frac{1 + (m + 1)\tilde{\kappa}_1}{|\tilde{\kappa}|}.$$

The domain  $D'_{(1)}$  that we have to study is then of the form

$$D'_{(1)} = \{(x_1, x_2) : 0 < x_1 < \varepsilon, |x_2 - \psi(x_1) - c_0 x_1^a| \leq \varepsilon x_1^a\},$$

where  $\psi(x_1) + c_0 x_1^a = \psi^{(2)}(x_1)$ . Moreover,

$$(3.3) \quad \phi^{(2)}(x_1, x_2) = \phi^a(x_1, x_2 - c_0 x_1^a) =: \tilde{\phi}^a(x_1, x_2),$$

so that  $\tilde{\phi}^a$  represents  $\phi$  in the modified adapted coordinates

$$(3.4) \quad y_1 := x_1, \quad y_2 := x_1 - \psi(x_1) - c_0 x_1^a,$$

compared to the adapted coordinates  $y_1 := x_1, \quad y_2 := x_1 - \psi(x_1)$ , in which  $\phi$  is represented by  $\phi^a$ .

Notice that the exponent  $a$  may be non-integer (but rational), so that  $\psi^{(2)}$  is in general only fractionally smooth, i.e., a smooth function of  $x_2$  and some fractional power of  $x_1$  only. The same applies to every  $\psi^{(l)}$  with  $l \geq 2$ , whereas  $\phi^a$  is still smooth, i.e., when we express  $\phi$  in our adapted coordinates, we still get a smooth function, whereas when we pass to modified adapted coordinates, we may only get fractionally smooth functions.

We shall write  $D^a$  for the domain  $D'_{(1)}{}^a$ , i.e.,

$$D^a := \{(y_1, y_2) : 0 < y_1 < \varepsilon, |y_2| < \varepsilon y_1^a\},$$

so that  $D^a$  represents our domain  $D'_{(1)}$  in our modified adapted coordinates, in which  $\phi$  is represented by  $\tilde{\phi}^a$ .

We assume that we are in Case 2, so that  $\partial_2 \tilde{\phi}^a_{\tilde{\kappa}}(1, 0) = 0$  and  $\partial_1 \tilde{\phi}^a_{\tilde{\kappa}}(1, 0) \neq 0$ .

We choose  $B \geq 2$  minimal so that  $\partial_2^B \tilde{\phi}^a_{\tilde{\kappa}}(1, 0) \neq 0$ . Since  $\tilde{\phi}^a_{\tilde{\kappa}}$  is  $\tilde{\kappa}$ -homogeneous, the principal part of  $\tilde{\phi}^a$  is then of the form (cf. (9.6) in [21])

$$(3.5) \quad \tilde{\phi}^a_{\tilde{\kappa}}(y_1, y_2) = y_2^B Q(y_1, y_2) + c_1 y_1^n, \quad c_1 \neq 0, \quad Q(1, 0) \neq 0,$$

where  $Q$  is a  $\tilde{\kappa}$ -homogeneous smooth function. Note that  $n$  is rational, but not necessarily integer, since we are in modified adapted coordinates.

Observe also that this implies that we may write

$$(3.6) \quad \tilde{\phi}^a(y_1, y_2) = y_2^B b_B(y_1, y_2) + y_1^n \alpha(y_1) + \sum_{j=1}^{B-1} y_2^j b_j(y_1),$$

with smooth functions  $b_B, \alpha$  such that  $\alpha(0) \neq 0$  and

$$b_B(y_1, y_2) = Q(y_1, y_2) + \text{terms of } \tilde{\kappa}\text{-degree strictly bigger than that of } Q,$$

and smooth functions  $b_1, \dots, b_{B-1}$  of  $y_1$ , which are either flat, or of finite type  $b_j(y_1) = y_1^{n_j} \alpha_j(y_1)$ , with smooth functions  $\alpha_j$  such that  $\alpha_j(0) \neq 0$ .

For convenience, we shall also write  $b_j(y_1) = y_1^{n_j} \alpha_j(y_1)$  when  $b_j$  is flat, keeping in mind that in this case we may choose  $n_j \in \mathbb{N}$  as large as we please (but  $\alpha_j(0) = 0$ ).

Notice that then, for  $j = 1, \dots, B-1$ ,  $y_2^j b_j(y_1)$  consists of terms of  $\tilde{\kappa}$ -degree strictly bigger than 1.

Recall that the Newton diagram of the  $\tilde{\kappa}$ -principal part  $\tilde{\phi}^a_{\tilde{\kappa}}$  is the line segment  $\gamma'_{(1)} = [(A'_{(0)}, B'_{(0)}), (A'_{(1)}, B'_{(1)})]$ , which must then contain a point with second coordinate given by  $B$ . It then follows easily that the following relations hold true:

$$\tilde{\kappa}_2 < 1, \quad 2 \leq m < \frac{1}{\tilde{\kappa}_1} = n, \quad B \leq B'_{(0)}.$$

Actually, since  $\tilde{\kappa}_2 B'_{(0)} \leq 1$ , we even have

$$(3.7) \quad \tilde{\kappa}_2 B \leq 1, \quad m < a = \frac{\tilde{\kappa}_2}{\tilde{\kappa}_1} \leq \frac{n}{B}.$$

As in [21], we define normalized measures  $\nu_k$  corresponding to the  $\mu_k$  by

$$\langle \nu_k, f \rangle := \int f\left(x_1, 2^{(m\tilde{\kappa}_1 - \tilde{\kappa}_2)k} x_2 + x_1^m \omega(2^{-k\tilde{\kappa}_1} x_1), \phi_k(x)\right) \eta(x) dx,$$

where again  $\eta$  is a smooth function with  $\text{supp } \eta \subset \{x_1 \sim 1, |x_2| < \varepsilon\}$  (for some small  $\varepsilon > 0$ ) and  $\phi_k(x_1, x_2) := 2^k \tilde{\phi}^a(2^{-\tilde{\kappa}_1 k} x_1, 2^{-\tilde{\kappa}_2 k} x_2)$  is given by

$$\phi_k(x_1, x_2) := x_2^B (Q(x) + O(2^{-\varepsilon' k})) + x_1^n \alpha(2^{-\tilde{\kappa}_1 k} x_1) + \sum_{j=1}^{B-1} x_2^j 2^{(1-j\tilde{\kappa}_2)k} b_j(2^{-\tilde{\kappa}_1 k} x_1)$$

for some  $\varepsilon' > 0$ . Observe that

$$2^{(1-j\tilde{\kappa}_2)k} b_j(2^{-\tilde{\kappa}_1 k} x_1) = x_1^{n_j} 2^{-(j\tilde{\kappa}_2 + n_j\tilde{\kappa}_1 - 1)k} \alpha_j(2^{-\tilde{\kappa}_1 k} x_1),$$

where  $(j\tilde{\kappa}_2 + n_j\tilde{\kappa}_1 - 1) > 0$ .

We write  $\nu_k$  as  $\nu_\delta$ , by putting

$$(3.8) \quad \langle \nu_\delta, f \rangle := \int f(x_1, \delta_0 x_2 + x_1^m \omega(\delta_1 x_1), \phi_\delta(x)) \eta(x) dx,$$

where  $\phi_\delta$  is of the form

$$(3.9) \quad \phi_\delta(x) := x_2^B b(x_1, x_2, \delta) + x_1^n \alpha(\delta_1 x_1) + r(x_1, x_2, \delta),$$

with

$$(3.10) \quad r(x_1, x_2, \delta) := \sum_{j=1}^{B-1} \delta_{j+2} x_2^j x_1^{n_j} \alpha_j(\delta_1 x_1),$$

and  $\delta = (\delta_0, \delta_1, \delta_2, \delta_3, \dots, \delta_{B+1})$  is given by

$$(3.11) \quad \delta := (2^{-k(\tilde{\kappa}_2 - m\tilde{\kappa}_1)}, 2^{-k\tilde{\kappa}_1}, 2^{-k\tilde{\kappa}_2}, 2^{-(n_1\tilde{\kappa}_1 + \tilde{\kappa}_2 - 1)k}, \dots, 2^{-(n_{B-1}\tilde{\kappa}_1 + (B-1)\tilde{\kappa}_2 - 1)k}).$$

Recall that  $\alpha(0) \neq 0$ , and that either  $\alpha_j(0) \neq 0$ , and then  $n_j$  is fixed (the type of the finite type function  $b_j$ ), or  $\alpha_j(0) = 0$ , and then we may assume that  $n_j$  is as large as we please.

Observe that  $\delta \rightarrow 0$  as  $k \rightarrow \infty$ , that every  $\delta_j$  is a power of  $\delta_0$ ,

$$\delta_j = \delta_0^{q_j}, \quad j = 1, \dots, B+1,$$

with positive exponents  $q_j > 0$  which are fixed rational numbers, except for those  $j \geq 2$  for which  $\alpha_{j-2}(0) = 0$ , for which we may choose the exponents  $q_j$  as large as we please.

Moreover,  $b(x_1, x_2, \delta)$  is a smooth function of all three arguments, and

$$(3.12) \quad b(x_1, x_2, 0) = Q(x_1, x_2).$$

For  $\delta$  sufficiently small, this implies in particular that  $b(x_1, 0, \delta) \neq 0$  when  $x_1 \sim 1$  and  $|x_2| < \varepsilon$ .

Assume we can prove that

$$(3.13) \quad \left( \int |\hat{f}|^2 d\nu_\delta \right)^{\frac{1}{2}} \leq C \|f\|_{L^{p_c}},$$



with  $C$  independent of  $\delta$ . Then straight-forward re-scaling by means of the  $\tilde{\kappa}$ -dilations leads to the estimate

$$(3.14) \quad \left( \int |\hat{f}|^2 d\mu_k \right)^{\frac{1}{2}} \leq C 2^{-k \frac{|\tilde{\kappa}|}{2}} \left( 1 - \frac{2(h_{\text{pr}} + 1)}{p'_c} \right) \|f\|_{L^{p_c}},$$

where  $p'_c \geq 2(h_{\text{pr}} + 1)$  (cf. 11.5) in [21]). So, our goal is to verify (3.13).

Observe also that the  $\tilde{\kappa}$ -principal parts of  $\tilde{\phi}^a$  and  $\phi_\delta$  do agree.

Recall that  $B \geq 2$ , and  $d \geq 2$ . We shall often use the interpolation parameter  $\theta_c := 2/p'_c = 1/(h^r + 1)$ . Since, by definition,  $h^r \geq d$ , the second assumption implies

$$(3.15) \quad \theta_c \leq \frac{1}{3}.$$

We first derive some useful estimates from below for  $p'_c = 2(h^r + 1)$ . We put  $H := 1/\tilde{\kappa}_2$ , so that

$$(3.16) \quad n = 1/\tilde{\kappa}_1, \quad H = 1/\tilde{\kappa}_2.$$

Note that  $H$  is rational, but not necessarily entire. We next define

$$\tilde{h}^r := \frac{mH}{m+1}, \quad \tilde{p}'_c := 2(\tilde{h}^r + 1), \quad \tilde{\theta}_c := \frac{2}{\tilde{p}'_c} = \frac{m+1}{mH + m + 1} \leq \tilde{\theta}_B := \frac{m+1}{mB + m + 1}.$$

Let us also put  $\tilde{p}'_B := 2/\tilde{\theta}_B \leq \tilde{p}'_c$ , and

$$p'_H := \frac{12H}{3+H}, \quad \theta_H := \frac{2}{p'_H} = \frac{1}{2H} + \frac{1}{6},$$

and define  $p'_B, \theta_B$  accordingly, with  $H$  replaced by  $B$ .

**Lemma 3.1.** (a) *We have  $p'_c > \tilde{p}'_c$ , unless  $h^r = \tilde{h}^r = d$  and  $h^r + 1 \geq H$ . In the latter case,  $p'_c = \tilde{p}'_c = 2(d+1)$ .*

(b) *If  $m \geq 3$  and  $H \geq 2$ , or  $m = 2$  and  $H \geq 3$ , then*

$$\tilde{p}'_c \geq p'_H \geq p'_B,$$

*where the inequality  $\tilde{p}'_c \geq p'_H$  is even strict unless  $m = 2$  and  $H = 3$ .*

*Proof.* (a) The Newton polyhedron  $\mathcal{N}(\tilde{\phi}^a)$  of  $\tilde{\phi}^a$  is contained in the closed half-space bounded from below by the principal line of  $\tilde{\phi}^a$ , which passes through the points  $(0, H)$  and  $(n, 0)$ . Moreover, it is known that the principal line  $L$  of  $\phi$  is a supporting line to  $\mathcal{N}(\tilde{\phi}^a)$  (this follows from Varchenko's algorithm), and it has slope  $1/m$ . It is therefore parallel to the line  $\tilde{L}$  passing through the points  $(0, H)$  and  $(mH, 0)$  and lies "above"  $\tilde{L}$  (see Figure 1). Thus the second coordinate  $d+1$  of the point of intersection of  $L$

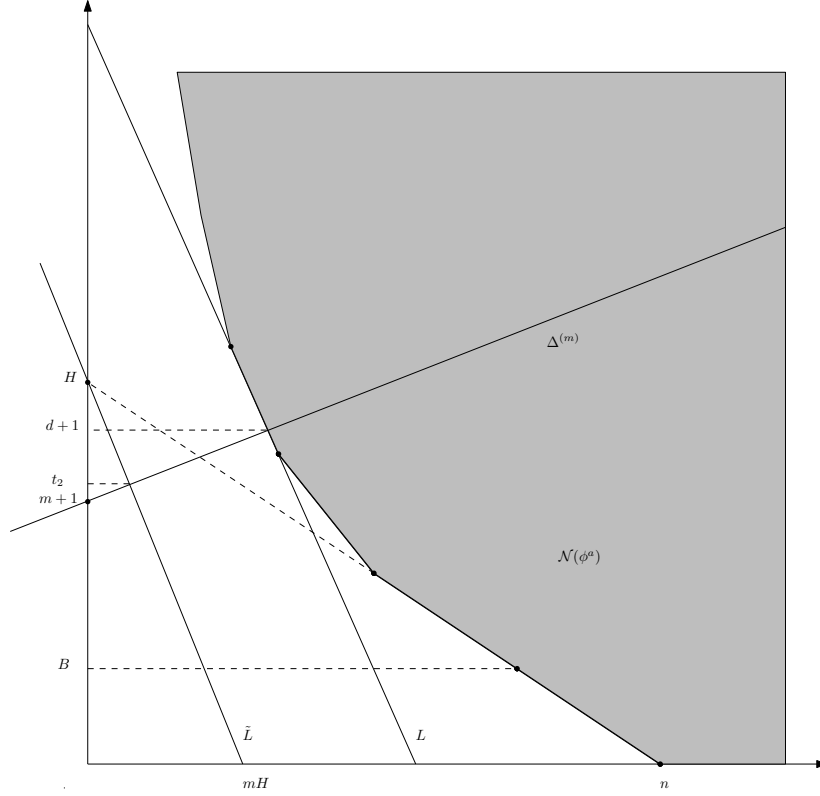


FIGURE 1

with  $\Delta^{(m)}$  is greater or equal to the second coordinate  $t_2$  of the point of intersection  $(t_1, t_2)$  of  $\tilde{L}$  with  $\Delta^{(m)}$ , so that  $h^r \geq d \geq t_2 - 1$ .

But, the point  $(t_1, t_2)$  is determined by the equations  $t_2 = m + 1 + t_1$  and  $t_2 = H - t_1/m$ , so that  $t_2 = (mH + m + 1)/(m + 1)$ . This shows that  $h^r \geq \tilde{h}^r$ , hence  $p'_c \geq \tilde{p}'_c$ .

Notice also that  $d + 1 > t_2$ , hence  $p'_c > \tilde{p}'_c$ , unless  $L = \tilde{L}$ .

So, assume that  $L = \tilde{L}$ . Then  $d = 1/(1/H + 1/mH) = \tilde{h}^r$ , and the principal face  $\pi(\tilde{\phi}^a)$  of  $\tilde{\phi}^a$  must be the edge  $[(0, H), (n, 0)]$  (see Figure 1). Thus, if  $h^r + 1 \geq H$ , then clearly  $h^r = d = \tilde{h}^r$ , and  $p'_c = \tilde{p}'_c$ . And, if  $h^r + 1 < H$ , then we see that  $h^r + 1$  is the second coordinate of the point of intersection of  $\Delta^{(m)}$  with  $\pi(\tilde{\phi}^a)$ , and thus

$$\tilde{h}^r + 1 < h^r + 1 = h_{l_{\text{pr}}} + 1 = \frac{1 + (m + 1)\tilde{\kappa}_1}{|\tilde{\kappa}|}$$

(cf. (3.2)).

(b) The inequality  $\tilde{p}'_c \geq p'_H$  is equivalent to

$$mH^2 - (2m + 5)H + 3m + 3 \geq 0,$$

so that the remaining statements are elementary to check.

Q.E.D.

The following corollary is a straight-forward consequence of the definition of  $\theta_H$  and Lemma 3.1.

**Corollary 3.2.** (a) *If  $m \geq 3$  and  $H \geq 2$ , or  $m = 2$  and  $H \geq 3$ , then  $\theta_c < \theta_B$ , unless  $m = 2$  and  $H = B = 3$  (where  $\theta_c = \theta_B = 1/3$ ).*  
 (b) *If  $h^r + 1 \leq B$ , then  $\theta_c < \tilde{\theta}_c$ , unless  $B = H = h^r + 1 = d + 1$ , where  $\theta_c = \tilde{\theta}_c$ .*  
 (c) *If  $H \geq 3$ , then  $\theta_c < 1/3$ , unless  $H = 3$  and  $m = 2$ .*

Recall next that the complete phase corresponding to  $\phi_\delta$  has the form

$$\Phi(x, \delta, \xi) := \xi_1 x_1 + \xi_2 (\delta_0 x_2 + x_1^m \omega(\delta_1 x_1)) + \xi_3 \phi_\delta(x_1, x_2),$$

where

$$\phi_\delta(x) = x_2^B b(x_1, x_2, \delta) + x_1^n \alpha(\delta_1 x_1) + r(x_1, x_2, \delta)$$

so that

$$\begin{aligned} \Phi(x, \delta, \xi) &= \xi_1 x_1 + \xi_2 x_1^m \omega(\delta_1 x_1) + \xi_3 x_1^n \alpha(\delta_1 x_1) \\ &+ \xi_2 \delta_0 x_2 + \xi_3 \left( x_2^B b(x_1, x_2, \delta) + r(x_1, x_2, \delta) \right). \end{aligned}$$

#### 4. SPECTRAL LOCALIZATION TO FREQUENCY BOXES WHERE $|\xi_i| \sim \lambda_i$ : THE CASE WHERE NOT ALL $\lambda_i$ 'S ARE COMPARABLE

Denote by  $T_\delta$  the operator of convolution with  $\widehat{\nu}_\delta$ , where we recall that

$$\widehat{\nu}_\delta(\xi) = \int e^{-i\Phi(x, \delta, \xi)} \eta(x) dx.$$

In a next step, as in Section 5 of [21] (where  $2^{-j}$  plays the same role as  $\delta_0$  here), we decompose

$$\nu_\delta = \nu_k = \sum_{\lambda} \nu_\delta^\lambda,$$

where the sum is taken over all triples  $\lambda = (\lambda_1, \lambda_2, \lambda_3)$  of dyadic numbers  $\lambda_i \geq 1$ , and where  $\nu_k^\lambda$  is localized to frequencies  $\xi$  such that  $|\xi_i| \sim \lambda_i$ , if  $\lambda_i > 1$ , and  $|\xi_i| \lesssim 1$ , if  $\lambda_i = 1$ . The cases where  $\lambda_i = 1$  for at least one  $\lambda_i$  can be dealt with in the same way as the corresponding cases where  $\lambda_i = 2$ , and therefore we shall always assume in the sequel that

$$\lambda_i > 1. \quad i = 1, 2, 3.$$

The spectrally localized measure  $\nu_\delta^\lambda(x)$  is then given by

$$\widehat{\nu_\delta^\lambda}(\xi) := \chi_1\left(\frac{\xi_1}{\lambda_1}\right) \chi_1\left(\frac{\xi_2}{\lambda_2}\right) \chi_1\left(\frac{\xi_3}{\lambda_3}\right) \widehat{\nu}_\delta(\xi),$$

i.e.,

$$\begin{aligned} \nu_\delta^\lambda(x) &= \lambda_1 \lambda_2 \lambda_3 \int \check{\chi}_1\left(\lambda_1(x_1 - y_1)\right) \check{\chi}_1\left(\lambda_2(x_2 - \delta_0 y_2 - y_1^m \omega(\delta_1 y_1))\right) \\ (4.1) \quad &\check{\chi}_1\left(\lambda_3(x_3 - \phi_\delta(y))\right) \eta(y) dy, \end{aligned}$$

where  $\tilde{\chi}$  is the inverse Fourier transform. Recall also that

$$(4.2) \quad \text{supp } \eta \subset \{y_1 \sim 1, |y_2| < \varepsilon\}, \quad (\varepsilon \ll 1).$$

Arguing as in [21], by making use of the localizations given by the first and the third factor of the integrand, the integration in  $y_1$  and  $y_2$  (here we can apply van der Corput type estimates) yield

$$\|\nu_\delta^\lambda\|_\infty \lesssim \lambda_1 \lambda_2 \lambda_3 \lambda_1^{-1} \lambda_3^{-\frac{1}{B}}.$$

Similarly, the localizations given by the first and second factor imply

$$\|\nu_\delta^\lambda\|_\infty \lesssim \lambda_1 \lambda_2 \lambda_3 \lambda_1^{-1} (\lambda_2 \delta_0)^{-1},$$

and consequently

$$(4.3) \quad \|\nu_\delta^\lambda\|_\infty \lesssim \min\{\lambda_2 \lambda_3^{\frac{B-1}{B}}, \lambda_3 \delta_0^{-1}\}.$$

We have to distinguish various cases. Notice first that it is easy to see that the phase function  $\Phi$  has no critical point with respect to  $x_1$  if one of the components  $\lambda_i$  of  $\lambda$  is much bigger than the two others, so that integrations by parts yield that

$$\|\widehat{\nu_\delta^\lambda}\|_\infty \lesssim |\lambda|^{-N} \quad \text{for every } N \in \mathbb{N},$$

which easily implies estimates for the operator  $T_\delta^\lambda : \varphi \mapsto \varphi * \widehat{\nu_\delta^\lambda}$  which are better than needed. We may therefore concentrate on the following, remaining cases.

Recall the interpolation parameter  $\theta_c = 2/p'_c \leq \frac{1}{3}$ .

**1. Case:**  $\lambda_1 \sim \lambda_3$ ,  $\lambda_2 \ll \lambda_1$ . Applying first the method of stationary phase in  $x_1$ , and then van der Corput's lemma in  $x_2$ , we find that

$$(4.4) \quad \|\widehat{\nu_\delta^\lambda}\|_\infty \lesssim \lambda_1^{-\frac{1}{2} - \frac{1}{B}}.$$

By interpolation, using this estimate and the first one in (4.3), we obtain

$$\|T_\delta^\lambda\|_{p \rightarrow p'} \lesssim \lambda_1^{-\frac{1}{B} - \frac{1}{2} + \frac{3}{2}\theta} \lambda_2^\theta,$$

where  $\theta = 2/p'$ . Summation over all dyadic  $\lambda_2$  with  $\lambda_2 \ll \lambda_1$  yields

$$\sum_{\lambda_2 \ll \lambda_1} \|T_\delta^\lambda\|_{p \rightarrow p'} \lesssim \lambda_1^{-\frac{1}{B} - \frac{1}{2} + \frac{5}{2}\theta}.$$

Notice that for  $\theta := \theta_B$  we have

$$-\frac{1}{B} - \frac{1}{2} + \frac{5}{2}\theta = \frac{1}{4}\left(\frac{1}{B} - \frac{1}{3}\right) \leq 0, \quad \text{if } B \geq 3,$$

and that strict inequality holds when  $\theta < \theta_B$ . But, Corollary 3.2 (a) shows that if  $H \geq 3$ , then indeed  $\theta_c < \theta_B$ , unless  $m = 2$  and  $H = B = 3$ . Consequently, for  $\theta := \theta_c$  we can sum over all dyadic  $\lambda_1$  (unless  $H = B = 3$  and  $m = 2$ ) and obtain

$$(4.5) \quad \|T_\delta^I\|_{p_c \rightarrow p'_c} \lesssim \sum_{\lambda_1 \sim \lambda_3, \lambda_2 \ll \lambda_1} \|T_\delta^\lambda\|_{p_c \rightarrow p'_c} \lesssim 1,$$

where  $T_\delta^I := \sum_{\lambda_1 \sim \lambda_3, \lambda_2 \ll \lambda_1} T_\delta^\lambda$  denotes the contribution by the operators  $T_\delta^\lambda$  which arise in this case. The constant in this estimate does not depend on  $\delta$ .

If  $H = B = 3$  and  $m = 2$ , then we only get a uniform estimate

$$\sum_{\lambda_2 \ll \lambda_1} \|T_\delta^\lambda\|_{p_c \rightarrow p'_c} \lesssim 1.$$

Finally, assume that  $B = 2$ . Since we assume that  $\theta_c \leq 1/3$ , we than again find that

$$-\frac{1}{B} - \frac{1}{2} + \frac{5}{2}\theta_c \leq -1 + \frac{5}{2}\frac{1}{3} < 0,$$

so that (4.5) remains valid.

Let us return to the case where  $H = B = 3$  and  $m = 2$ , hence  $\theta_c = 1/3$ , which will require more refined methods.

In a first step, we shall take the sum of the  $\nu_\delta^\lambda$  over all dyadic  $\lambda_2 \ll \lambda_1$ . Moreover, since  $\lambda_1 \sim \lambda_3$ , we may reduce to the case where  $\lambda_3 = 2^M \lambda_1$ , where  $M \in \mathbb{N}$  is fixed and not too large. For the sake of simplicity of notation, we then assume that  $M = 0$ . All this then amounts to considering the functions  $\sigma_\delta^{\lambda_1}$  given by

$$\widehat{\sigma_\delta^{\lambda_1}}(\xi) = \chi_1\left(\frac{\xi_1}{\lambda_1}\right) \chi_0\left(\frac{\xi_2}{\lambda_1}\right) \chi_1\left(\frac{\xi_3}{\lambda_1}\right) \widehat{\nu}_\delta(\xi),$$

where now  $\chi_0$  is smooth, compactly supported in an interval  $[-\varepsilon, \varepsilon]$ , where  $\varepsilon > 0$  is sufficiently small, and  $\chi_0 \equiv 1$  in the interval  $[-\varepsilon/2, \varepsilon/2]$ . In particular,  $\sigma_\delta^{\lambda_1}(x)$  is given again by the expression (4.1), only with the second factor  $\check{\chi}_1\left(\lambda_2(x_2 - \delta_0 y_2 - y_1^m \omega(\delta_1 y_1))\right)$  in the integrand replaced by  $\check{\chi}_0\left(\lambda_1(x_2 - \delta_0 y_2 - y_1^m \omega(\delta_1 y_1))\right)$  and  $\lambda_2$  replaced by  $\lambda_1$ . Thus we obtain the same type of estimates as in (4.3), i.e.,

$$(4.6) \quad \|\widehat{\sigma_\delta^{\lambda_1}}\|_\infty \lesssim \lambda_1^{-\frac{5}{6}}, \quad \|\sigma_\delta^{\lambda_1}\|_\infty \lesssim \lambda_1^{\frac{2}{3}} \min\{\lambda_1, \delta_0^{-1} \lambda_1^{\frac{1}{3}}\} = \lambda_1 \min\{\lambda_1^{\frac{2}{3}}, \delta_0^{-1}\}.$$

By  $T_\delta^{\lambda_1}$  we shall denote the operator of convolution with  $\widehat{\sigma_\delta^{\lambda_1}}$ .

In view of (4.6), we shall distinguish between two subcases:

**1.1. The subcase where  $\lambda_1 \leq \delta_0^{-3/2}$ .** In this case, by (4.6) we have

$$(4.7) \quad \|\widehat{\sigma_\delta^{\lambda_1}}\|_\infty \lesssim \lambda_1^{-\frac{5}{6}}, \quad \|\sigma_\delta^{\lambda_1}\|_\infty \lesssim \lambda_1^{\frac{5}{3}},$$

so that

$$\|T_\delta^{\lambda_1}\|_{p_c \rightarrow p'_c} \lesssim \lambda_1^{-\frac{1}{3}} \lambda_1^{\frac{1}{3}} = 1,$$

and summing these estimates does not lead to the desired uniform estimate. Let us denote by

$$T_\delta^{I_1} := \sum_{\lambda_1 \leq \delta_0^{-3/2}} T_\delta^{\lambda_1}$$

the contribution by the operators  $T_\delta^\lambda$  which arise in this subcase. In order to prove the desired estimate

$$(4.8) \quad \|T_\delta^{I_1}\|_{p_c \rightarrow p'_c} \lesssim 1,$$

we shall therefore have to apply an interpolation argument (see Subsection 5.1).

**1.2. The subcase where  $\lambda_1 > \delta_0^{-3/2}$ .** In this case we have  $\|\sigma_\delta^{\lambda_1}\|_\infty \lesssim \delta_0^{-1}\lambda_1$ , and interpolation yields

$$\|T_\delta^{\lambda_1}\|_{p_c \rightarrow p'_c} \lesssim \delta_0^{-\frac{1}{3}} \lambda_1^{-\frac{2}{9}}.$$

If we denote by  $T_\delta^{I_2} := \sum_{\lambda_1 > \delta_0^{-3/2}} T_\delta^{\lambda_1}$  the contribution by the operators  $T_\delta^\lambda$  which arise in this subcase, we thus obtain

$$(4.9) \quad \|T_\delta^{I_2}\|_{p_c \rightarrow p'_c} \lesssim \sum_{\lambda_1 > \delta_0^{-3/2}} \delta_0^{-\frac{1}{3}} \lambda_1^{-\frac{2}{9}} \lesssim 1.$$

**2. Case:  $\lambda_2 \sim \lambda_3$  and  $\lambda_1 \ll \lambda_2$ .** Here, we can estimate  $\widehat{\nu_\delta^\lambda}$  in the same way as in the previous case and obtain  $\|\widehat{\nu_\delta^\lambda}\|_\infty \lesssim \lambda_2^{-1/2} \lambda_3^{-1/B} \sim \lambda_2^{-1/2-1/B}$ . Moreover, by (4.3), we have  $\|\nu_\delta^\lambda\|_\infty \lesssim \lambda_2 \min\{\lambda_2^{(B-1)/B}, \delta_0^{-1}\}$ . Both these estimates are independent of  $\lambda_1$ . Assuming here without loss of generality that  $\lambda_2 = \lambda_3$ , we therefore consider the sum over all  $\nu_\delta^\lambda$  such that  $\lambda_1 \ll 1$ , by putting  $\sigma_\delta^{\lambda_2} := \sum_{\lambda_1 \ll \lambda_2} \nu_\delta^{(\lambda_1, \lambda_2, \lambda_2)}$ . This means that

$$\widehat{\sigma_\delta^{\lambda_2}}(\xi) = \chi_0\left(\frac{\xi_1}{\lambda_2}\right) \chi_1\left(\frac{\xi_2}{\lambda_2}\right) \chi_1\left(\frac{\xi_3}{\lambda_2}\right) \widehat{\nu_\delta}(\xi),$$

where now  $\chi_0$  is smooth and compactly supported in an interval  $[-\varepsilon, \varepsilon]$ , where  $\varepsilon > 0$  is sufficiently small. In particular,  $\sigma_\delta^{\lambda_2}(x)$  is given again by the expression (4.1), only with the first factor  $\check{\chi}_1(\lambda_1(x_1 - y_1))$  in the integrand replaced by  $\check{\chi}_0(\lambda_2(x_1 - y_1))$  and  $\lambda_1$  replaced by  $\lambda_2$ . Thus we obtain the same type of estimates

$$(4.10) \quad \|\widehat{\sigma_\delta^{\lambda_2}}\|_\infty \lesssim \lambda_2^{-\frac{1}{2}-\frac{1}{B}}, \quad \|\sigma_\delta^{\lambda_2}\|_\infty \lesssim \lambda_2 \min\{\lambda_2^{\frac{B-1}{B}}, \delta_0^{-1}\}.$$

Denote by  $T_\delta^{\lambda_2}$  the operator of convolution with  $\widehat{\sigma_\delta^{\lambda_2}}$ .

Interpolating between the first estimate in (4.10) and the estimate  $\|\sigma_\delta^{\lambda_2}\|_\infty \lesssim \lambda_2 \lambda_2^{(B-1)/B}$ , we get

$$\|T_\delta^{\lambda_2}\|_{p \rightarrow p'} \lesssim \lambda_2^{-\frac{1}{B}-\frac{1}{2}+\frac{5}{2}\theta}.$$

Arguing as in the previous case, we see that this still suffices to sum over all dyadic  $\lambda_2$  for  $\theta = \theta_c = 2/p'_c$ , to obtain the desired estimate

$$(4.11) \quad \|T_\delta^{II}\|_{p_c \rightarrow p'_c} \lesssim \sum_{\lambda_2} \|T_\delta^{\lambda_2}\|_{p_c \rightarrow p'_c} \lesssim 1,$$

unless  $H = B = 3$  and  $m = 2$ . Here  $T_\delta^{II}$  denotes the contribution by the operators  $T_\delta^\lambda$  which arise in this case.

So, assume that  $H = B = 3$  and  $m = 2$ , so that  $\theta_c = 1/3$ . Then we distinguish two subcases:

**2.1. The subcase where  $\lambda_2 \leq \delta_0^{-B/(B-1)} = \delta_0^{-3/2}$ .** In this case, (4.10) reads

$$(4.12) \quad \|\widehat{\sigma_\delta^{\lambda_2}}\|_\infty \lesssim \lambda_2^{-\frac{5}{6}}, \quad \|\sigma_\delta^{\lambda_2}\|_\infty \lesssim \lambda_2^{\frac{5}{3}},$$

which implies our previous estimate

$$\|T_\delta^{\lambda_2}\|_{p_c \rightarrow p'_c} \lesssim 1.$$

Let us denote by

$$T_\delta^{II_1} := \sum_{\lambda_2 \leq \delta_0^{-3/2}} T_\delta^{\lambda_2}$$

the contribution by the operators  $T_\delta^\lambda$  which arise in this subcase. In order to prove the desired estimate

$$(4.13) \quad \|T_\delta^{II_1}\|_{p_c \rightarrow p'_c} \lesssim 1,$$

we shall thus have to apply an interpolation argument once more (see Subsection 5.1).

**2.2. The subcase where  $\lambda_2 > \delta_0^{-B/(B-1)} = \delta_0^{-3/2}$ .** Then (4.10) implies that  $\|\sigma_\delta^{\lambda_2}\|_\infty \lesssim \lambda_2 \delta_0^{-1}$ , hence

$$\|T_\delta^{\lambda_2}\|_{p_c \rightarrow p'_c} \lesssim \delta_0^{-\theta_c} \lambda_2^{(\frac{3}{2} + \frac{1}{B})\theta_c - \frac{1}{2} - \frac{1}{B}} = \delta_0^{-\frac{1}{3}} \lambda_2^{-\frac{2}{9}}.$$

As in Subcase 1.2, this implies the desired estimate

$$(4.14) \quad \|T_\delta^{II_2}\|_{p_c \rightarrow p'_c} \lesssim 1$$

for the contributions  $T_\delta^{II_2}$  of the operators  $T_\delta^\lambda$  with  $\lambda$  satisfying the assumptions of this subcase to  $T_\delta$ .

**3. Case:  $\lambda_1 \sim \lambda_2$  and  $\lambda_3 \ll \lambda_1$ .** If  $\lambda_3 \ll \lambda_2 \delta_0$ , then the phase function has no critical point in  $x_2$ , and so an integrations by part in  $x_2$  and the stationary phase method in  $x_1$  yield

$$\|\widehat{\nu_\delta^\lambda}\|_\infty \lesssim \lambda_1^{-\frac{1}{2}} (\lambda_2 \delta_0)^{-N} \lesssim \lambda_2^{-\frac{1}{2}} \lambda_3^{-N}$$

for every  $N \in \mathbb{N}$ , and the second estimate in (4.3) implies that

$$\|\nu_\delta^\lambda\|_\infty \lesssim \lambda_3 \delta_0^{-1}.$$

Interpolating these estimates we obtain

$$\|T_\delta^\lambda\|_{p \rightarrow p'} \lesssim \lambda_3^{-N'} \lambda_2^{-\frac{1}{2}(1-\theta)} \delta_0^{-\theta},$$

where  $N'$  can be chosen arbitrarily large if  $\theta < 1$ . But, if  $\theta = \theta_c$ , then  $\theta \leq 1/3$ , and since  $\lambda_2 \delta_0 \geq 1$  if  $\lambda_3 \ll \lambda_2 \delta_0$ , we see that

$$\sum_{\lambda_1 \sim \lambda_2, \lambda_3 \ll \lambda_2 \delta_0} \|T_\delta^\lambda\|_{p_c \rightarrow p'_c} \lesssim \delta_0^{\frac{1-3\theta}{2}} \lesssim 1.$$

Let us therefore assume from now on that in addition  $\lambda_3 \gtrsim \lambda_2 \delta_0$ . Then we can first apply the method of stationary phase to the integration in  $x_1$  and subsequently van der Corput's estimate to the  $x_2$ -integration and obtain the estimate

$$(4.15) \quad \|\widehat{\nu_\delta^\lambda}\|_\infty \lesssim \lambda_2^{-\frac{1}{2}} \lambda_3^{-\frac{1}{B}}.$$

In view of (4.3), we distinguish two subcases.

**3.1 The subcase where  $\lambda_3^{1/B} > \lambda_2 \delta_0$ .** Then interpolation of the first estimate in (4.3) with (4.15) yields

$$\|T_\delta^\lambda\|_{p \rightarrow p'} \lesssim \lambda_2^{\frac{3\theta-1}{2}} \lambda_3^{\theta-\frac{1}{B}}.$$

Since  $\theta_c \leq 1/3$ , we have  $3\theta_c - 1 \leq 0$  (even with strict inequality, unless  $B = 2$ , or  $H = B = 3$  and  $m = 2$ , because of Corollary 3.2 (c)). If  $3\theta_c - 1 < 0$ , we can sum over all dyadic  $\lambda_2 \gg \lambda_3$  and obtain

$$\sum_{\{\lambda_2: \lambda_3 \ll \lambda_2 < \delta_0^{-1} \lambda_3^{\frac{1}{B}}\}} \|T_\delta^\lambda\|_{p_c \rightarrow p'_c} \lesssim \lambda_3^{-\frac{1}{B} - \frac{1}{2} + \frac{5}{2}\theta}.$$

If

$$T_\delta^{III_1} := \sum_{\lambda_1 \sim \lambda_2, \lambda_3 \ll \delta_0^{-\frac{B}{B-1}}, \lambda_3 \ll \lambda_2 < \delta_0^{-1} \lambda_3^{\frac{1}{B}}} T_\delta^\lambda$$

denotes the contribution by the operators  $T_\delta^\lambda$  which arise in this subcase, this implies in a similar way as before that

$$(4.16) \quad \|T_\delta^{III_1}\|_{p_c \rightarrow p'_c} \lesssim 1.$$

If  $H = B = 3$  and  $m = 2$ , then we only get a uniform estimate

$$\|T_\delta^\lambda\|_{p_c \rightarrow p'_c} \lesssim 1,$$

and in order to establish (4.16), we shall apply an interpolation argument in Subsection 5.2.

If  $B = 2$  and  $\theta_c = 1/3$  (hence  $m = 2$ ), then we can first sum over the dyadic  $\lambda_3$  with  $\lambda_3 > (\lambda_2 \delta_0)^2$ , provided  $\lambda_2 \delta_0 \gtrsim 1$ , because  $\theta_c - 1/B = -1/6$ , and then we sum over the  $\lambda_2$  for which  $\lambda_2 \delta_0 \gtrsim 1$ . So, in order to (4.16) in this case, we are left with the estimation of the operator

$$T_\delta^{III_0} := \sum_{\lambda_1 \sim \lambda_2, \lambda_3 \ll \delta_0^{-1}, \lambda_3 \ll \lambda_2 \ll \delta_0^{-1}} T_\delta^\lambda.$$

This will also be done by means of complex interpolation in Subsection 5.2.

**3.2 The subcase where  $\lambda_3^{1/B} \leq \lambda_2 \delta_0$ .** Assuming without loss of generality (in a similar way as before) that  $\lambda_1 = \lambda_2$ , then the second estimate in (4.3) implies that  $\|\nu_\delta^\lambda\|_\infty \lesssim \lambda_3 \delta_0^{-1}$ , and in combination with (4.15) we obtain

$$(4.17) \quad \|T_\delta^\lambda\|_{p_c \rightarrow p'_c} \lesssim \lambda_2^{-\frac{1-\theta}{2}} \lambda_3^{\frac{(B+1)\theta-1}{B}} \delta_0^{-\theta},$$



where  $\theta := \theta_c$ . Notice that in this subcase

$$\lambda_3 \leq \min\{(\lambda_2 \delta_0)^B, \lambda_2\}, \quad \text{and} \quad \lambda_2 \delta_0 \lesssim \lambda_3.$$

In view of this, we shall distinguish two cases:

(a)  $\lambda_2 \geq \delta_0^{-\frac{B}{B-1}}$ . Assume first that  $B \geq 3$ . Then

$$(B+1)\theta_B - 1 = \frac{B^2 - 2B + 3}{6B} > 0$$

for  $B \geq 2$ , and since  $\theta = \theta_c \leq \theta_B = 1/(2B) + 1/6$  by Corollary 3.2, we see that we can sum the estimates in (4.17) over all dyadic  $\lambda_3 \ll \lambda_2$  and get

$$\sum_{\lambda_3 \ll \lambda_2} \|T_\delta^\lambda\|_{p_c \rightarrow p'_c} \lesssim \lambda_2^{\frac{(3B+2)\theta - B - 2}{2B}} \delta_0^{-\theta},$$

Using again that  $\theta \leq \theta_B$ , we find that for  $B \geq 2$ ,

$$(3B+2)\theta - B - 2 \leq (3B+2)\left(\frac{1}{6} + \frac{1}{2B}\right) - B - 2 = \frac{-3B^2 - B + 6}{6B} < 0,$$

so we can sum also in  $\lambda_2 \geq \delta_0^{-\frac{B}{B-1}}$  and find that  $\|T_\delta^{III_{2a}}\|_{p_c \rightarrow p'_c} \lesssim \delta_0^{-\frac{5B\theta - B - 2}{2(B-1)}}$ , where  $T_\delta^{III_{2a}} := \sum_{\lambda_1 \sim \lambda_2 \geq \delta_0^{-\frac{B}{B-1}}, \lambda_3 \ll \lambda_2} T_\delta^\lambda$ . But,

$$(4.18) \quad 5B\theta - B - 2 \leq 5B\theta_B - B - 2 = 5B\left(\frac{1}{2B} + \frac{1}{6}\right) - B - 2 = \frac{3-B}{6} \leq 0,$$

if  $B \geq 3$ , and thus for  $B \geq 3$  we get

$$(4.19) \quad \|T_\delta^{III_{2a}}\|_{p_c \rightarrow p'_c} \lesssim 1.$$

The remains the case  $B = 2$ . Here, for  $\theta = \theta_c$ , we have  $(B+1)\theta - 1 = 3\theta - 1 \leq 0$ , and

$$\|T_\delta^\lambda\|_{p_c \rightarrow p'_c} \lesssim \lambda_2^{-\frac{1-\theta}{2}} \lambda_3^{\frac{3\theta-1}{2}} \delta_0^{-\theta}$$

Assume first that  $\theta_c < 1/3$ . Then we can first sum in  $\lambda_3 \geq \lambda_2 \delta_0$  (notice that  $\lambda_2 \delta_0 > 1$ ) and obtain

$$\sum_{\lambda_3 \geq \lambda_2 \delta_0} \|T_\delta^\lambda\|_{p_c \rightarrow p'_c} \lesssim \lambda_2^{2\theta-1} \delta_0^{\frac{\theta-1}{2}}.$$

Then we sum over  $\lambda_2 \geq \delta_0^{-1}$  and get an estimate by  $C\delta_0^{\frac{1-3\theta}{2}} \lesssim 1$ , so that (4.19) remains true also in this case.

Assume finally that  $\theta_c = 1/3$ . Then  $\|T_\delta^\lambda\|_{p_c \rightarrow p'_c} \lesssim \lambda_2^{-\frac{1}{3}} \delta_0^{-\frac{1}{3}}$ . Summing first in  $\lambda_3 \ll \lambda_2$ , we get an estimate by  $C(\log \lambda_2) \lambda_2^{-\frac{1}{3}} \delta_0^{-\frac{1}{3}}$ , and summation over all  $\lambda_2 \geq \delta_0^{-B/(B-1)} = \delta_0^{-2}$  leads to an estimate of order  $\log(1/\delta_0) \delta_0^{1/3} \lesssim 1$ . Thus, again (4.19) holds true.

(b)  $\lambda_2 < \delta_0^{-\frac{B}{B-1}}$ . Then we use  $\lambda_3 \leq (\lambda_2 \delta_0)^B$ , i.e.,  $\lambda_2 \geq \lambda_3^{1/B} \delta_0^{-1}$ , and summation of the estimate (4.17) over these  $\lambda_2$  yields

$$\sum_{\{\lambda_2: \lambda_2 \geq \lambda_3^{1/B} \delta_0^{-1}\}} \|T_\delta^\lambda\|_{p_c \rightarrow p'_c} \lesssim \lambda_3^{\frac{(3+2B)\theta-3}{2B}} \delta_0^{\frac{1-3\theta}{2}}.$$

If the exponent of  $\lambda_3$  on the right-hand side of this estimate is strictly negative, then we see that

$$(4.20) \quad \|T_\delta^{III_{2b}}\|_{p_c \rightarrow p'_c} \lesssim 1,$$

where

$$T_\delta^{III_{2b}} := \sum_{\lambda_1 \sim \lambda_2 < \delta_0^{-\frac{B}{B-1}}, \lambda_3 \ll (\lambda_2 \delta_0)^B} T_\delta^\lambda.$$

So, assume that the exponent is non-negative, and notice that our assumptions in this case imply that  $\lambda_3 \leq \delta_0^{-B/(B-1)}$ . Summation over all these dyadic  $\lambda_3$  then leads to

$$\begin{aligned} \sum_{\lambda_2 \geq \lambda_3^{1/B} \delta_0^{-1}, \lambda_3 \leq \delta_0^{-B/(B-1)}} \|T_\delta^\lambda\|_{p_c \rightarrow p'_c} &\lesssim (\log \frac{1}{\delta_0}) (\delta_0^{-B/(B-1)})^{\frac{(3+2B)\theta-3}{2B}} \delta_0^{\frac{1-3\theta}{2}} \\ &= (\log \frac{1}{\delta_0}) \delta_0^{-\frac{5B\theta-B-2}{2(B-1)}}. \end{aligned}$$

However, if we assume without loss of generality that  $\theta = \theta_B$ , then we have seen in (4.18) that  $5B\theta - B - 2 \leq 0$ , if  $B \geq 3$ . Moreover, by Corollary 3.2 (a) we know that  $\theta_c < \theta_B$ , unless  $B = H = 3$  and  $m = 2$ . Thus, using  $\theta = \theta_c$  in place of  $\theta_B$ , when  $B \geq 3$  we have  $5B\theta_c - B - 2 \leq 0$ , and we again obtain (4.20), unless  $B = H = 3$  and  $m = 2$ . Note that in the latter case  $\theta_c = 1/3$ , so that

$$\sum_{\{\lambda_2: \lambda_2 \geq \lambda_3^{1/3} \delta_0^{-1}\}} \|T_\delta^\lambda\|_{p_c \rightarrow p'_c} \lesssim 1.$$

The proof of (4.20) in this particular case, where

$$T_\delta^{III_{2b}} = \sum_{\delta_0^{-1} \lambda_3^{1/3} \ll \lambda_1 \sim \lambda_2 < \delta_0^{-3/2}} T_\delta^\lambda,$$

will therefore again require the use of some interpolation argument, in order to control the summation in  $\lambda_3$ . We remark that in this case, estimate (4.17) reads as  $\|T_\delta^\lambda\|_{p_c \rightarrow p'_c} \lesssim \lambda_1^{-1/3} \lambda_3^{1/9} \delta_0^{-1/3}$ , and

$$\sum_{1 \leq \lambda_3 \lesssim \lambda_1 \delta_0} \sum_{\lambda_1 < \delta_0^{-2/3}} \lambda_1^{-1/3} \lambda_3^{1/9} \delta_0^{-1/3} \lesssim 1,$$

since  $\lambda_1 \delta_0 \geq 1$  in this double sum.

This shows that if  $B = H = 3$  and  $m = 2$  (hence  $\theta_c = 1/3$ ), what remains to be estimated is the operator

$$T_\delta^{II_2} := \sum_{\delta_0^{-1} \ll \lambda_1 < \delta_0^{-3/2}} \sum_{\lambda_1 \delta_0 \ll \lambda_3 \ll (\lambda_1 \delta_0)^3} T_\delta^\lambda,$$

This will be done in Subsection 5.4.

Assume finally that  $B = 2$ . The case where  $\theta_c < 1/3$  can be treated as in the previous case (a), so assume that  $\theta_c = 1/3$ . Then  $\|T_\delta^\lambda\|_{p_c \rightarrow p'_c} \lesssim (\lambda_2 \delta_0)^{-\frac{1}{3}}$ . Summing first over all  $\lambda_2$  such that  $\lambda_2 \delta_0 \geq \lambda_3^{1/2}$  leads to an estimate of order  $\lambda_3^{-1/6}$ , which then allows to sum also in  $\lambda_3$ . Thus, again (4.20) holds true.

## 5. INTERPOLATION ARGUMENTS FOR THE OPEN CASES WHERE $m = 2$ AND $B = 3$ OR $B = 2$

Let us assume in this section that  $m = 2$  and  $B = 3$  or  $B = 2$ . Our goal will be to establish the estimates (4.8), (4.13), (4.16) and (4.20) for the operators  $T_\delta^{I_1}, T_\delta^{II_1}, T_\delta^{III_1}$  and  $T_\delta^{III_0}$ , as well as  $T_\delta^{III_{2b}}$ , also at the endpoint  $p_c$  corresponding to  $\theta_c = 1/3$  (hence  $p'_c = 2/\theta_c = 6$ ). These cases had been left open in the previous section.

The estimates for the operators  $T_\delta^{III_1}, T_\delta^{III_0}$  and  $T_\delta^{III_{2b}}$  will be established by means of complex interpolation, roughly in analogy with the proof of Proposition 5.1 (a) and (b) in [21].

Also the operators  $T_\delta^{I_1}$  and  $T_\delta^{II_1}$  could be handled by means of complex interpolation. However, a shorter proof is possible by means of the real interpolation approach developed by Bak and Seeger in [4] (which, however, requires the validity of the expected estimates of the operators  $T_\delta^{I_1}, T_\delta^{II_1}$ , as well as of several further operators).

**5.1. Estimation of  $T_\delta^{I_1}$  and  $T_\delta^{II_1}$ : Real interpolation.** The estimation of the operators  $T_\delta^{I_1}$  and  $T_\delta^{II_1}$  will follow the same scheme, so let us consider  $T_\delta^{I_1}$  only, which is the operator of convolution with  $\widehat{\sigma_\delta^{I_1}}$ , where  $\sigma_\delta^{I_1}$  denotes the measure

$$\sigma_\delta^{I_1} := \sum_{0 \leq j \leq \delta_0^{-3/2}} \sigma_\delta^{2^j}.$$

In the following discussion, if  $\mu$  is any bounded, complex Borel measure on  $\mathbb{R}^d$ , we shall often denote by  $T_\mu$  the convolution operator

$$T_\mu : \varphi \mapsto \varphi * \hat{\mu}.$$

Regretfully, we cannot apply Theorem 1.1 in [4] directly to the measure  $\mu = \sigma_\delta^{I_1}$ , because an essential assumption in [4] is that  $\mu$  is a bounded positive measure, and  $\sigma_\delta^{I_1}$  will not be positive. However, the method of proof in [4] easily yields the following variant of Theorem 1.1, which can be applied in our situation:

**Proposition 5.1.** *Let  $\mu$  be a positive Borel measure on  $\mathbb{R}^d$  of total mass  $\|\mu\|_1 \leq 1$ , and let  $p_0 \in [1, 2[$ . Assume that  $\mu$  can be decomposed into a finite sum*

$$\mu = \mu^b + \sum_{i \in I} \mu^i$$

*of bounded complex Borel measures  $\mu^b$  and  $\mu^i, i \in I$ , such that the following hold true: There is a constant  $A \geq 0$  such that:*

(a) *The operator  $T_{\mu^b}$  is bounded from  $L^{p_0}(\mathbb{R}^d)$  to  $L^{p'_0}(\mathbb{R}^d)$ , with*

$$(5.1) \quad \|T_{\mu^b}\|_{p_0 \rightarrow p'_0} \leq A.$$

(b) *Each of the measures  $\mu^i$  decomposes as*

$$\mu^i = \sum_{j=1}^{K_i} \mu_j^i = \sum_{j=1}^{K_i} \mu * \phi_j^i,$$

*where  $K_i \in \mathbb{N} \cup \{\infty\}$ , and where the  $\phi_j^i$  are integrable functions such that*

$$(5.2) \quad \|\phi_j^i\|_1 \leq 1.$$

*Moreover, there are constants  $a_i > 0, b_i > 0$  such that for all  $i$  and  $j$ ,*

$$(5.3) \quad \|\mu_j^i\|_\infty \leq A 2^{ja_i};$$

$$(5.4) \quad \|\widehat{\mu_j^i}\|_\infty \leq A 2^{-jb_i};$$

$$(5.5) \quad p_0 = 2 \frac{a_i + b_i}{2a_i + b_i} \text{ ( i.e., if } \theta a_i - (1 - \theta)b_i = 0, \text{ then } \frac{1}{p_0} = \frac{\theta}{2} + (1 - \theta)).$$

*Then there is a constant  $C$  which depends only on  $d$  and any compact interval in  $]0, \infty[$  containing the  $a_i$  and  $b_i$  such that for every  $i$ ,*

$$(5.6) \quad \|T_{\mu^i} f\|_{L^{p'_0}} \leq CA \|f\|_{L^{p_0}},$$

*and consequently*

$$(5.7) \quad \int |\hat{f}|^2 d\mu \leq CA \|f\|_{L^{p_0}(\mathbb{R}^d)}^2,$$

*Proof.* By essentially following the proof of Proposition 2.1 in [4], we define the interpolation parameter  $\theta := 2/p'_0$ . Observe that by (5.5) we have  $\theta = b_i/(a_i + b_i)$ , hence  $(1 - \theta)(-b_i) + \theta a_i = 0$  for every  $i$ . Thus, the two inequalities (5.3) and (5.4) allow to apply an interpolation trick due to Bourgain [5] and to conclude that each of the operators  $T_{\mu_j^i}$  is of restricted weak-type  $(p_0, p'_0)$ , with operator norm  $\leq CA$ , and if  $J$  is any compact subinterval of  $]0, \infty[$ , then for  $a_i, b_i \in J$  we may chose the constant  $C$  so that it depends only on  $J$ . In combination with (5.1), this implies that also  $T_\mu$  is of restricted weak-type  $(p_0, p'_0)$ , with operator norm  $\leq CA$ , where  $C$  may be different

from the previous constant, but with similar properties. By applying Tomas'  $R^*R$ -argument for the restriction operator  $R$ , we get

$$\int |\hat{f}|^2 d\mu \leq CA \|f\|_{L^{p_0,1}}^2.$$

In combination with Plancherel's theorem and (5.3) and (5.2) we can next use this estimate as in [4] to control

$$\|T_{\mu_j^i} f\|_2 = \|f * \widehat{\mu_j^i}\|_2 \leq A^{\frac{1}{2}} 2^{j\frac{a_i}{2}} \|\phi_j^i\|_1^{\frac{1}{2}} A^{\frac{1}{2}} \|f\|_{L^{p_0,1}}^2 \leq A 2^{j\frac{a_i}{2}} \|f\|_{L^{p_0,1}}^2.$$

It is here where the positivity of the measure  $\mu$  is used in an essential way. The remaining part of the argument in [4] does not require positivity of the underlying measure, so that it applies to each of the complex measures  $\mu^i$  as well, and we may conclude that for any  $s \in ]0, \infty]$ ,

$$\|T_{\mu^i} f\|_{L^{p'_0,s}} \leq CA \|f\|_{L^{p_0,s}}$$

(compare Proposition 2.1, (2.2), in [4]), where  $L^{p,s}$  denotes the Lorentz space of type  $p, s$ . Choosing  $s = 2$ , so that  $p_0 \leq 2 \leq p'_0$ , by the nesting properties of the scale of Lorentz spaces this implies in particular that

$$\|T_{\mu^i} f\|_{L^{p'_0,p'_0}} \leq CA \|f\|_{L^{p_0,p_0}},$$

hence (5.6). The same type of estimate then holds for the operator  $T_\mu$ , and Tomas' argument then leads to (5.7).

Q.E.D.

Let us return to our measure  $\nu_\delta$ . This is a positive measure, so we can choose  $\mu := \nu_\delta$  in Proposition 5.1. The spectral decompositions of the measure  $\nu_\delta$  in Section 4 as well as Section 6 amounts to a decomposition of the measure  $\nu_\delta$  into a finite sum of complex measures

$$(5.8) \quad \nu_\delta = \sum_{j \in J} \nu^j + \sum_{i \in I} \mu^i,$$

in such a way, that the convolution operators  $T_{\nu^j}$  corresponding to the measure  $\nu^j$  ( $j \in J$ ) from the first class will be bounded from  $L^{p_c}$  to  $L^{p'_c}$ , whereas the measure  $\mu^i$ , ( $i \in I$ ) from the second class will satisfy the conditions required on the measures  $\mu^i$  of Proposition 5.1. For instance, the operators  $T_\delta^{I_2}$ ,  $T_\delta^{II_2}$  and  $T_\delta^{III_{2a}}$  from Section 4 belong to the first class (compare the estimates (4.9), (4.14), (4.19)), but also  $T_\delta^{III_1}$  and  $T_\delta^{III_2}$  (the corresponding estimates (4.16) and (4.20) will be established by means of complex interpolation in the next subsection), whereas the operators  $T_\delta^{I_1}$  and  $T_\delta^{II_1}$  will belong to the second class.

We may then put  $\mu^b := \sum_{j \in J} \nu^j$  in Proposition 5.1. Let us show for instance that the measure  $\mu^i := \sigma_\delta^{I_1}$  corresponding to the operator  $T_\delta^{I_1}$  satisfies the assumptions of this proposition:

Recall to this end that  $\mu_j^i := \sigma_\delta^{2^j} = \nu_\delta * \phi_j$ , where the Fourier transform of  $\phi_j$  is given by  $\widehat{\phi_j}(\xi) = \chi_1\left(\frac{\xi_1}{\lambda_1}\right)\chi_0\left(\frac{\xi_2}{\lambda_1}\right)\chi_1\left(\frac{\xi_3}{\lambda_1}\right)$ . This implies a uniform estimate of the  $L^1$ -norms of the  $\phi_j$  of the form (5.2) (possibly not with constant 1, but a fixed constant, which does not matter). Moreover, the estimates (5.3) and (5.4) are satisfied because of (4.7), with exponents  $a_i := 5/3$  and  $b_i := 5/6$ , so that  $p_0 = 6/5 = p_c$ ,

Similar arguments apply also to the measure  $\mu^i := \sigma_\delta^{II_1}$  corresponding to the operator  $T_\delta^{II_1}$ , where the exponents  $a_i$  and  $b_i$  will be the same (compare (4.12)), as well as to the other measures of the first class which will appear later.

**5.2. Estimation of  $T_\delta^{III_1}$ : Complex interpolation.** The discussion of this operator will somewhat resemble the one of the operator  $T_{\delta,j}^V$  in Subsection 8.1 of [21], which arose from the same Subcase 3.1 of Subsection 5.3 in [21], with  $2^{-j}$  playing the role of  $\delta_0$  here, and where we have had  $B = 2$ , in place of  $B = 3$  here. We shall make use of a more refined structural result for the phase function  $\phi_\delta$  which will be derived in a more general context in Section 8 (compare (8.2)). In view of this result, in combination with Corollary 8.1, we may and shall assume that

$$(5.9) \quad \phi_\delta(x) := x_2^3 b(x_1, x_2, \delta) + x_1^n \alpha(\delta_1 x_1) + \delta_3 x_2 x_1^{n_1} \alpha_1(\delta_1 x_1), \quad (x_1 \sim 1, |x_2| < \varepsilon),$$

where we may now assume (compare (8.3)) that

$$(5.10) \quad b(x_1, x_2, \delta) = b_3(\delta_1 x_1, \delta_2 x_2), \quad b_3(0, 0) = 1.$$

Moreover,  $\alpha(0) \neq 0$ , and either  $\alpha_1(0) \neq 0$ , and then  $n_1$  is fixed, or  $\alpha_1(0) = 0$ , and then we may assume that  $n_1$  is as large as we please (notice also that by (8.5),  $\delta_3 = 2^{-k(n_1 \tilde{\kappa}_1 + \tilde{\kappa}_2 - 1)}$  is coupled with  $n_1$ ), so that in particular in this case  $\delta_3 \ll \delta_0$ . Observe also that  $\delta_2 \ll \delta_0$ .

Useful tools will also be the Lemmas 7.2 and 8.1 from [21] on oscillatory sums and double sums. For the convenience of the reader, let us at least recall the first lemma (in the sharper version of Remark 7.3 of [21]):

**Lemma 5.2.** *Let  $Q = \prod_{j=1}^n [-R_k, R_k] \subset \mathbb{R}^n$  be a compact cuboid, with  $R_k > 0, k = 1, \dots, n$ , and let  $H$  be a  $C^1$ -function on an open neighborhood of  $Q$ . Moreover, let  $\alpha, \beta^1, \dots, \beta^n \in \mathbb{R}^\times$  be given. For any given real numbers  $a_1, \dots, a_n \in \mathbb{R}^\times$  and  $M \in \mathbb{N}$  we then put*

$$(5.11) \quad F(t) := \sum_{l=0}^M 2^{i \alpha l t} (H \chi_Q) \left( 2^{\beta^1 l} a_1, \dots, 2^{\beta^n l} a_n \right).$$

Assume that there are constants  $\epsilon \in ]0, 1]$  and  $C_k, k = 1, \dots, n$ , such that

$$(5.12) \quad \int_0^1 \left| \frac{\partial H}{\partial u_k}(su) \right| ds \leq C_k |u_k|^{\epsilon-1}, \quad \text{for all } u \in Q.$$

Then there is a constant  $C$  depending on  $Q$ , the numbers  $\alpha$  and  $\beta^k$  and  $\epsilon$ , but not on  $H$ , the  $a_k$ ,  $M$  and  $t$ , such that

$$(5.13) \quad |F(t)| \leq C \frac{|H(0)| + \sum_k C_k}{|2^{i\alpha t} - 1|}, \quad \text{for all } t \in \mathbb{R}, a_1, \dots, a_2 \in \mathbb{R}^\times \text{ and } M \in \mathbb{N}.$$

In particular, we have

$$|F(t)| \leq C \frac{\|H\|_{C^1(Q)}}{|2^{i\alpha t} - 1|}, \quad \text{for all } t \in \mathbb{R}, a_1, \dots, a_2 \in \mathbb{R}^\times \text{ and } M \in \mathbb{N}.$$

Coming back to the operator  $T_\delta^{III_1}$ , observe that  $\lambda_1 \sim \lambda_2$  in the definition of  $T_\delta^{III_1}$ , so that we may and shall assume without loss of generality that  $\lambda_1 = \lambda_2$ . In order to verify estimate (4.16), we then have to prove

**Proposition 5.3.** *Let  $m = 2$  and  $B = 3$ , and consider the measure*

$$\nu_\delta^{III_1} := \sum_{2^M \leq \lambda_3 \leq 2^{-M} \delta_0^{-3/2}} \sum_{2^M \lambda_3 \leq \lambda_1 \leq \delta_0^{-1} \lambda_3^{1/3}} \nu_\delta^{(\lambda_1, \lambda_1, \lambda_3)},$$

where summation is taken over all sufficiently large dyadic  $\lambda_i \geq 2^M$  in the given range. If we denote by  $T_\delta^{III_1}$  the operator of convolution with  $\widehat{\nu_\delta^{III_1}}$ , then, if  $M \in \mathbb{N}$  is sufficiently large (and  $\epsilon$  sufficiently small),

$$(5.14) \quad \|T_\delta^{III_1}\|_{6/5 \rightarrow 6} \leq C,$$

with a constant  $C$  not depending on  $\delta$ , for  $\delta$  sufficiently small.

*Proof.* Recall that, by (4.15),  $\|\widehat{\nu_\delta^\lambda}\|_\infty \lesssim \lambda_1^{-1/2} \lambda_3^{-1/3}$ . We therefore define here for  $\zeta$  in the strip  $\Sigma = \{\zeta \in \mathbb{C} : 0 \leq \operatorname{Re} \zeta \leq 1\}$  an analytic family of measures by

$$\mu_\zeta(x) := \gamma(\zeta) \sum_{2^M \leq 2^{k_3} \leq 2^{-M} \delta_0^{-\frac{3}{2}}} \sum_{2^{M+k_3} \leq 2^{k_1} \leq \delta_0^{-1} 2^{\frac{k_3}{3}}} 2^{\frac{(1-3\zeta)k_1}{2}} 2^{\frac{(1-3\zeta)k_3}{3}} \nu_\delta^{(2^{k_1}, 2^{k_1}, 2^{k_3})},$$

where  $\gamma(\zeta)$  is an entire function which will serve a similar role as the function  $\gamma(z)$  in the proof of Proposition 5.2(a) in [21]. We shall choose  $\gamma(\zeta) = \gamma_1(\zeta)\gamma_2(\zeta)\gamma_3(\zeta)$  as the product of three factors  $\gamma_j(\zeta)$ , whose precise definition will be given in the course of the proof. It will be uniformly bounded on  $\Sigma$ , and such that  $\gamma(\theta_c) = \gamma(1/3) = 1$ .

By  $T_\zeta$  we denote the operator of convolution with  $\widehat{\mu_\zeta}$ . Observe that for  $\zeta = \theta_c = 1/3$ , we have  $\mu_{\theta_c} = \nu_\delta^{III_1}$ , hence  $T_{\theta_c} = T_\delta^{III_1}$ , so that, again by Stein's interpolation theorem for analytic families of operators, (5.14) will follow if we can prove the following estimates on the boundaries of the strip  $\Sigma$ :

$$\begin{aligned} \|\widehat{\mu_{it}}\|_\infty &\leq C & \forall t \in \mathbb{R}, \\ \|\mu_{1+it}\|_\infty &\leq C & \forall t \in \mathbb{R}. \end{aligned}$$

The first estimate is an immediate consequence of our estimate for  $\widehat{\nu_\delta^\lambda}$ , since these functions have essentially disjoint supports, so let us concentrate on the second estimate, i.e., assume that  $\zeta = 1 + it$ , with  $t \in \mathbb{R}$ . We then have to prove that there is constant  $C$  such that

$$(5.15) \quad |\mu_{1+it}(x)| \leq C,$$

where  $C$  is independent of  $t, x$  and  $\delta$ .

Let us introduce the measures  $\mu_{\lambda_1, \lambda_3}$  given by

$$\mu_{\lambda_1, \lambda_3}(x) := \lambda_1^{-1} \lambda_3^{-\frac{2}{3}} \nu_\delta^{(\lambda_1, \lambda_1, \lambda_3)}(x),$$

which allow to re-write

$$(5.16) \quad \mu_{1+it}(x) = \gamma(1+it) \sum_{2^M \leq \lambda_3 \leq 2^{-M} \delta_0^{-\frac{2}{3}}} \sum_{2^M \lambda_3 \leq \lambda_1 \leq \delta_0^{-1} \lambda_3^{\frac{1}{3}}} \lambda_1^{-\frac{3}{2}it} \lambda_3^{-it} \mu_{\lambda_1, \lambda_3}(x).$$

Recall also from (4.1) (in combination with (5.9)) that

$$(5.17) \quad \begin{aligned} \mu_{\lambda_1, \lambda_3}(x) &= \lambda_1 \lambda_3^{\frac{1}{3}} \int \check{\chi}_1 \left( \lambda_1(x_1 - y_1) \right) \check{\chi}_1 \left( \lambda_1(x_2 - \delta_0 y_2 - y_1^2 \omega(\delta_1 y_1)) \right) \\ &\quad \check{\chi}_1 \left( \lambda_3 \left( x_3 - y_2^3 b(y_1, y_2, \delta) - y_1^n \alpha(\delta_1 y_1) - \delta_3 y_2 y_1^{n_1} \alpha_1(\delta_1 y_1) \right) \right) \eta(y) dy, \end{aligned}$$

where  $\eta$  is supported where  $y_1 \sim 1$  and  $|y_2| < \varepsilon$ . Assume first that  $|x_1| \gg 1$ , or  $|x_1| \ll 1$ . Since  $\check{\chi}_1$  is rapidly decreasing, and  $\lambda_3 \ll \lambda_1$ , we easily see that  $|\mu_{\lambda_1, \lambda_3}(x)| \leq C_N \lambda_1^{-N} \lambda_3^{-N}$  for every  $N \in \mathbb{N}$ , which immediately implies (5.15). A similar argument applies if  $|x_2| \gg 1$ . However, if  $|x_1| + |x_2| \lesssim 1$  and  $|x_3| \gg 1$ , we can only conclude (after scaling by  $1/\lambda_1$  in  $y_1$ ) that  $|\mu_{\lambda_1, \lambda_3}(x)| \leq C_N \lambda_3^{-N}$ , which allows to sum in  $\lambda_3$ , but the summation in  $\lambda_1$  remains a problem.

Let us thus assume from now on that  $|x_1| \sim 1$  and  $|x_2| \lesssim 1$ .

By means of the change of variables  $y_1 \mapsto x_1 - y_1/\lambda_1$ ,  $y_2 \mapsto y_2/\lambda_3^{1/3}$  and Taylor expansion around  $x_1$  we may re-write

$$(5.18) \quad \mu_{\lambda_1, \lambda_3}(x) = \iint \check{\chi}_1(y_1) F_\delta(\lambda_1, \lambda_3, x, y_1, y_2) dy_1 dy_2,$$

where

$$\begin{aligned} F_\delta(\lambda_1, \lambda_3, x, y_1, y_2) &:= \eta(x_1 - \lambda_1^{-1} y_1, \lambda_3^{-\frac{1}{3}} y_2) \check{\chi}_1(D - E y_2 + r_1(y_1)) \\ &\times \check{\chi}_1 \left( A - B y_2 - y_2^3 b(x_1 - \lambda_3^{-1} (\lambda_3 \lambda_1^{-1} y_1), \lambda_3^{-\frac{1}{3}} y_2, \delta) + \lambda_3 \lambda_1^{-1} \left( r_2(y_1) + (\lambda_3^{-\frac{1}{3}} y_2) \delta_3 r_3(y_1) \right) \right). \end{aligned}$$

Here, the quantities  $A$  to  $E$  are given by

$$(5.19) \quad \begin{aligned} A &= A(x, \lambda_3, \delta) := \lambda_3 Q_A(x), & B &:= B(x, \lambda_3, \delta) := \lambda_3^{\frac{2}{3}} Q_B(x), \\ D &= D(x, \lambda_1, \delta) := \lambda_1 Q_D(x), & E &= E(\lambda_1, \lambda_3, \delta) := \lambda_1 \lambda_3^{-\frac{1}{3}} \delta_0 \leq 1, \end{aligned}$$



with

$$Q_A(x) := x_3 - x_1^n \alpha(\delta_1 x_1), \quad Q_B(x) := \delta_3 x_1^{n_1} \alpha_1(\delta_1 x_1), \quad Q_D(x) := x_2 - x_1^2 \omega(\delta_1 x_1),$$

and do not depend on  $y_1, y_2$ . Moreover, the functions  $r_i(y_1) = r_i(y_1; \lambda_1^{-1}, x_1, \delta)$ ,  $i = 1, 2, 3$ , are smooth functions of  $y_1$  (and  $\lambda_1^{-1}$  and  $x_1$ ) satisfying estimates of the form (5.20)

$$|r_i(y_1)| \leq C|y_1|, \quad \left| \left( \frac{\partial}{\partial(\lambda_1^{-1})} \right)^l r_i(y_1; \lambda_1^{-1}, x_1, \delta) \right| \leq C_l |y_1|^{l+1} \quad \text{for every } l \geq 1.$$

Notice that we may here assume that  $|y_1| \lesssim \lambda_1$ , because of our assumption  $|x_1| \lesssim 1$  and the support properties of  $\eta$ . It will also be important to observe that  $E = \delta_0 \lambda_1 \lambda_3^{-\frac{1}{3}} \leq 1$  for the index set of  $\lambda_1, \lambda_3$  over which we sum in (5.16). Notice also that  $|\lambda_3^{-1/3} y_2| \leq \varepsilon$ .

Let us choose  $c > 0$  so that  $|r_i(y_1)| \leq c(1 + |y_1|)$ ,  $i = 1, 2, 3$ .

In order to verify (5.15), given  $x$ , we shall split the sum in (5.16) into four parts, which will be treated subsequently in different ways (compare the analogous discussion in Subsection 8.1 of [21]).

**1. The part where  $\max\{|A|, |B|\} \geq 1$  and  $|D| \geq 4c$ .** Denote by  $\mu_{1+it}^1(x)$  the contribution to  $\mu_{1+it}(x)$  by the terms for which  $\max\{|A(x, \lambda_3, \delta)|, |B(x, \lambda_3, \delta)|\} \geq 1$  and  $|D(x, \lambda_1, \delta)| \geq 4c$ .

We claim that here

$$(5.21) \quad |\mu_{\lambda_1, \lambda_3}(x)| \lesssim |D|^{-\frac{1}{4}} \max\{|A|^{\frac{1}{3}}, |B|^{\frac{1}{2}}\}^{-\frac{1}{4}}.$$

In view of (5.19), this estimate will allow us to sum over all dyadic  $\lambda_1, \lambda_3$  for which the corresponding quantities  $A, B$  and  $D$  satisfy the conditions of this subcase, and we obtain the right kind of estimate  $|\mu_{1+it}^1(x)| \leq C$ , in agreement with (5.15).

In order to prove (5.21), let us first consider the contribution  $\mu_{\lambda_1, \lambda_3}^1(x)$  to the integral defining  $\mu_{\lambda_1, \lambda_3}(x)$  by the region where  $|y_1| > |D|/4c$ . Here we may estimate  $|\check{\chi}_1(y_1)| \lesssim |D|^{-N}$  for every  $N \in \mathbb{N}$ . Moreover, if  $|x_3| \gg 1$ , then  $|A| \gg \lambda_3 \gg |B|^{3/2}$ , and  $A$  becomes the dominant term in the argument of the last factor of  $F_\delta$ . Therefore we may estimate

$$|\mu_{\lambda_1, \lambda_3}(x)| \leq C_N |D|^{-N} \lambda_3^{\frac{1}{3}} |A|^{-N}$$

for every  $N \in \mathbb{N}$ , which is stronger than (5.21).

And, if  $|x_3| \lesssim 1$ , then we may apply Lemma 14.1 (with  $T := \lambda_3^{1/3}$ ,  $\epsilon := 0$ ) and obtain the estimate

$$|\mu_{\lambda_1, \lambda_3}^1(x)| \lesssim |D|^{-N} \max\{|A|^{\frac{1}{3}}, |B|^{\frac{1}{2}}\}^{-\frac{1}{2}},$$

which is still stronger than required in (5.21).

Denote next by  $\mu_{\lambda_1, \lambda_3}^2(x)$  the contribution by the region where  $|y_1| \leq |D|/4c$ . Then  $|r_i(y_1)| \leq |D|/2$ , and thus if in addition  $|Ey_2| \leq |D|/4$ , or  $|Ey_2| > 2|D|$ , then  $|D -$

$Ey_2 + r_1(y_1) \Big| \geq |D|/4$ . We may then estimate the second factor of  $F_\delta$  by  $C_N |D|^{-N}$ , which allows to argue as before. So, let us assume that  $|r_1(y_1)| \leq |D|/2$  and  $|D|/4 \leq |Ey_2| \leq 2|D|$ . Then  $|y_2| \sim |D|/|E| \geq |D| \gtrsim 1$ . In case that  $|D| \lesssim \max\{|A|^{1/3}, |B|^{1/2}\}$ , we may apply Lemma 14.1 (assuming again that  $|x_3| \lesssim 1$ ; the other case is again easier) and find that

$$|\mu_{\lambda_1, \lambda_3}^2(x)| \lesssim \max\{|A|^{1/3}, |B|^{1/2}\}^{-1/2} \leq |D|^{-1/4} \max\{|A|^{1/3}, |B|^{1/2}\}^{-1/4},$$

as desired. So assume that  $|D| \gg \max\{|A|^{1/3}, |B|^{1/2}\}$ . Then one easily sees that

$$\left| A - By_2 - y_2^3 b(x_1 - \lambda_1^{-1} y_1, \lambda_3^{-1/3} y_2, \delta) + \lambda_3 \lambda_1^{-1} \left( r_2(y_1) + (\lambda_3^{-1/3} y_2) \delta_3 r_3(y_1) \right) \right| \gtrsim |y_2| D^2,$$

so that

$$|\mu_{\lambda_1, \lambda_3}^2(x)| \lesssim \iint (1 + |y_1|)^{-N} (1 + |y_2| D^2)^{-N} dy_1 dy_2 \lesssim D^{-2} \lesssim |D|^{-1} \max\{|A|^{1/3}, |B|^{1/2}\}^{-1},$$

which is again stronger then required in (5.21).

**2. The part where  $\max\{|A|, |B|\} \geq 1$  and  $|D| < 4c$ .** Denote by  $\mu_{1+it}^2(x)$  the contribution to  $\mu_{1+it}(x)$  by the terms for which  $\max\{|A(x, \lambda_3, \delta)|, |B(x, \lambda_3, \delta)|\} \geq 1$  and  $|D(x, \lambda_1, \delta)| < 4c$ .

Let us fix  $\lambda_3$  satisfying  $2^M \leq \lambda_3 \leq 2^{-M} \delta_0^{-3/2}$  and  $\max\{|A(x, \lambda_3, \delta)|, |B(x, \lambda_3, \delta)|\} \geq 1$  in the first sum in (5.16). In order to compute  $\mu_{1+it}^2(x)$ , we then have to study the following sum in  $\lambda_1 = 2^{k_1}$ :

$$\sigma^2(\lambda_3, t, x) := \sum_{\{\lambda_1: 2^M \lambda_3 \leq \lambda_1 \leq \delta_0^{-1} \lambda_3^{1/3}, \lambda_1 |Q_D(x)| < 4c\}} \lambda_1^{-\frac{3}{2}it} \mu_{\lambda_1, \lambda_3}(x).$$

Indeed, we have  $\mu_{1+it}^2(x) = \sum_{\lambda_3} \lambda_3^{-it} \sigma^2(\lambda_3, t, x)$ , where summation is over all dyadic  $\lambda_3$  in the range described before.

The oscillatory sum defining  $\sigma^2(\lambda_3, t, x)$  can essentially be written in the form (5.11), with  $\alpha := -3/2, l = k_1$  and

$$u_1 = 2^{\beta^{1l}} a_1 := \lambda_1^{-1} \lambda_3, \quad u_2 = 2^{\beta^{2l}} a_2 := \lambda_1 Q_D(x), \quad u_3 = 2^{\beta^{3l}} a_3 := \lambda_1 (\lambda_3^{-1/3} \delta_0)$$

and where the function  $H = H_{\lambda_3, x, \delta}$  of  $u := (u_1, u_2, u_3)$  is given by

$$\begin{aligned} H(u) := & \iint \check{\chi}_1(y_1) \eta(x_1 - u_1 \lambda_3^{-1} y_1, \lambda_3^{-1/3} y_2) \check{\chi}_1(u_2 - u_3 y_2 + r_1(y_1; \lambda_3^{-1} u_1, x_1, \delta)) \\ & \times \check{\chi}_1 \left( A - By_2 - y_2^3 b(x_1 - \lambda_3^{-1} u_1 y_1, \lambda_3^{-1/3} y_2, \delta) + u_1 r_2(y_1; \lambda_3^{-1} u_1, x_1, \delta) \right. \\ & \left. + (\lambda_3^{-1/3} y_2) \delta_3 r_3(y_1; \lambda_3^{-1} u_1, x_1, \delta) \right) dy_1 dy_2. \end{aligned}$$

Moreover, the cuboid  $Q$  in Lemma 5.2 is defined by the conditions

$$|u_1| \leq 2^{-M}, \quad |u_2| < 4c, \quad |u_3| \leq 1.$$

Let us estimate the  $C^1$ - norm of  $H$  on  $Q$ . If  $|x_3| \gg 1$ , then  $|Q_A(x)| \gg 1$ , whereas  $|Q_B(x)| \lesssim 1$ , so that  $|A| \gg |B|$ , hence  $\max\{|A|, |B|\} = |A|$ . We even have that  $|By_2| \ll \lambda_3 |Q_B(x)| \ll |A|$ , as well as  $|y_2^3 b(x_1 - \lambda_3^{-1} u_1 y_1, \lambda_3^{-\frac{1}{3}} y_2, \delta)| \ll |A|$ . Making use also of the rapid decay of  $\check{\chi}_1(y_1)$ , this easily implies that

$$|H(u)| \lesssim |A|^{-N}$$

for every  $N \in \mathbb{N}$ , uniformly on  $Q$ .

On the other hand, if  $|x_3| \lesssim 1$ , then  $|A| \lesssim \lambda_3$ , so that the assumptions of Lemma 14.1 are satisfied (if we essentially put  $T := \lambda_3^{1/3}$ ), and we may conclude that

$$(5.22) \quad |H(u)| \lesssim \max\{|A|^{\frac{1}{3}}, |B|^{\frac{1}{2}}\}^{-\frac{1}{2}}, \quad \text{for all } u \in Q.$$

We have seen that this estimate holds true no matter which size  $|x_3|$  may have.

We next consider partial derivatives of  $H$ . From our integral formula for  $H(u)$ , it is obvious that the partial derivative of  $H$  with respect to  $u_1$  will essentially only produce additional factors of the form  $\lambda_3^{-1} y_1, \lambda_3^{-1} y_1 y_2^3, \lambda_3^{-1/3} y_2 \lambda_3^{-1} y_1$  under the double integral. However, powers of  $y_1$  can be absorbed by the rapidly decaying factor  $\check{\chi}_1(y_1)$ , and  $|\lambda_3^{-1} y_2^3| \leq \varepsilon \ll 1$ , so that  $|\partial_{u_1} H(u)|$  will satisfy an estimate of the form (5.22) as well. It is also easy to see that  $|\partial_{u_2} H(u)|$  satisfies such an estimate too.

More of a problem is the partial derivative of  $H$  with respect to  $u_3$ . This will essentially produce an additional factor  $y_2$  under the double integral. More precisely, let us put

$$g(y_1, y_2; u) := -\check{\chi}_1(y_1) y_2 \check{\chi}'_1(u_2 - u_3 y_2 + r_1(y_1; \lambda_3^{-1} u_1, x_1, \delta)),$$

so that

$$\begin{aligned} \partial_{u_3} H(u) &:= \iint \eta(x_1 - u_1 \lambda_3^{-1} y_1, \lambda_3^{-\frac{1}{3}} y_2) g(y_1, y_2; u) \\ &\quad \times \check{\chi}_1 \left( A - By_2 - y_2^3 b(x_1 - \lambda_3^{-1} u_1 y_1, \lambda_3^{-\frac{1}{3}} y_2, \delta) + u_1 r_2(y_1; \lambda_3^{-1} u_1, x_1, \delta) \right. \\ &\quad \left. + (\lambda_3^{-\frac{1}{3}} y_2) \delta_3 r_3(y_1; \lambda_3^{-1} u_1, x_1, \delta) \right) dy_1 dy_2. \end{aligned}$$

We claim that for every  $\epsilon \in ]0, 1]$  and  $s \in [0, 1]$  we have

$$(5.23) \quad |g(y_1, y_2; su)| \leq C_N |u_3|^{\epsilon-1} |y_2|^\epsilon s^{\epsilon-1} (1 + |y_1|)^{-N} \quad \text{for every } N \in \mathbb{N}$$

in the integrand. Indeed, if  $|su_3 y_2| \gg (1 + |y_1|)$ , then the third factor in  $g(y_1, y_2; u)$  can be estimated by  $C |su_3 y_2|^{\epsilon-1}$ , because of the rapid decay of  $\check{\chi}'_1$ , and (5.23) follows, and if  $|su_3 y_2| \lesssim (1 + |y_1|)$ , then  $|y_2| \lesssim |y_2|^\epsilon (|su_3|^{-1} (1 + |y_1|))^{1-\epsilon}$ , and (5.23) follows again.

By means of (5.23), we may now estimate

$$\begin{aligned} |\partial_{u_3} H(su)| &\lesssim |u_3|^{\epsilon-1} s^{\epsilon-1} \iint (1 + |y_1|)^{-N} \left( 1 + |A - By_2 - y_2^3 b(x_1 - \lambda_3^{-1} su_1 y_1, \lambda_3^{-\frac{1}{3}} y_2, \delta) \right. \\ &\quad \left. + su_1 r_2(y_1; \lambda_3^{-1} su_1, x_1, \delta) + (\lambda_3^{-\frac{1}{3}} y_2) \delta_3 r_3(y_1; \lambda_3^{-1} su_1, x_1, \delta) \right|^{-N} |y_2|^\epsilon dy_1 dy_2, \end{aligned}$$

and arguing from here on as before (distinguishing between the cases where  $|x_3| \gg 1$  and where  $|x_3| \lesssim 1$ ), we obtain by means of Lemma 14.1 that

$$|\partial_{u_3} H(su)| \lesssim |u_3|^{\epsilon-1} s^{\epsilon-1} \max\{|A|^{\frac{1}{3}}, |B|^{\frac{1}{2}}\}^{\epsilon-\frac{1}{2}}, \quad \text{for all } u \in Q.$$

This implies for every sufficiently small  $\epsilon > 0$  that for all  $u \in Q$ ,

$$\int_0^1 \left| \frac{\partial H}{\partial u_k}(su) \right| ds \leq C |u_k|^{\epsilon-1} \max\{|A|^{\frac{1}{3}}, |B|^{\frac{1}{2}}\}^{\epsilon-\frac{1}{2}}.$$

By means of Lemma 5.2, we may thus conclude that

$$|\sigma^2(\lambda_3, t, x)| \lesssim \frac{1}{|2^{-\frac{3}{2}it} - 1|} \max \left\{ (\lambda_3 |Q_A(x)|)^{\frac{1}{3}}, (\lambda_3^{\frac{2}{3}} |Q_B(x)|)^{\frac{1}{2}} \right\}^{\epsilon-\frac{1}{2}}.$$

Finally, this estimate allows to sum also in  $\lambda_3$ , and we conclude that  $|\mu_{1+it}^2(x)| \leq C$ , provided we choose the second factor in the definition of  $\gamma(\zeta)$  as  $\gamma_2(\zeta) := 2^{\frac{3}{2}(1-\zeta)} - 1$ .

**3. The part where  $\max\{|A|, |B|\} < 1$  and  $|D| \geq 4c$ .** Denote by  $\mu_{1+it}^3(x)$  the contribution to  $\mu_{1+it}(x)$  by the terms for which  $\max\{|A(x, \lambda_3, \delta)|, |B(x, \lambda_3, \delta)|\} < 1$  and  $|D(x, \lambda_1, \delta)| \geq 4c$ .

This case can be treated again by means of Lemma 5.2, only with the roles of  $\lambda_1$  and  $\lambda_3$  interchanged. So, let us here fix  $\lambda_1$  satisfying  $2^{2M} \leq \lambda_1 \leq 2^{-M/3} \delta_0^{-\frac{3}{2}}$  and  $\lambda_1 |Q_D(x)| \geq 4c$ , and consider the remaining sum in  $\lambda_3$  in (5.16), i.e.,

$$\sigma^3(\lambda_1, t, x) := \sum_{\{\lambda_3: \delta_0^3 \lambda_1^3 \leq \lambda_3 \leq 2^{-M} \lambda_1, \lambda_3 |Q_A(x)| < 1, \lambda_3^{\frac{2}{3}} |Q_B(x)| < 1\}} \lambda_3^{-it} \mu_{\lambda_1, \lambda_3}(x).$$

Notice that then  $\mu_{1+it}^3(x) = \sum_{\lambda_1} \lambda_1^{-3it/2} \sigma^3(\lambda_1, t, x)$ , where summation is over all dyadic  $\lambda_1$  in the range described before.

Also  $\sigma^3(\lambda_1, t, x)$  can essentially be written in the form (5.11), with  $\alpha := -1$  and  $l = k_3$  (if  $\lambda_3 = 2^{k_3}$ ), and

$$\begin{aligned} u_1 &= 2^{\beta^1 l} a_1 := \lambda_3 \lambda_1^{-1}, \quad u_2 = 2^{\beta^2 l} a_2 := \lambda_3^{-\frac{1}{3}}, \quad u_3 = 2^{\beta^3 l} a_3 := \lambda_3^{-\frac{1}{3}} (\lambda_1 \delta_0) \\ u_4 &= 2^{\beta^4 l} a_4 := \lambda_3 Q_A(x), \quad u_5 = 2^{\beta^5 l} a_5 := \lambda_3^{\frac{2}{3}} Q_B(x), \end{aligned}$$

and where the function  $H = H_{\lambda_1, x, \delta}$  of  $u := (u_1, \dots, u_5)$  is given by

$$\begin{aligned} H(u) &:= \iint \check{\chi}_1(y_1) \eta(x_1 - \lambda_1^{-1} y_1, u_2 y_2) \check{\chi}_1(D - u_3 y_2 + r_1(y_1; \lambda_1^{-1}, x_1, \delta)) \\ &\quad \times \check{\chi}_1 \left( u_4 - u_5 y_2 - y_2^3 b(x_1 - \lambda_1^{-1} y_1, u_2 y_2, \delta) + u_1 r_2(y_1; \lambda_1^{-1}, x_1, \delta) \right. \\ &\quad \left. + (u_2 y_2) \delta_3 r_3(y_1; \lambda_1^{-1}, x_1, \delta) \right) dy_1 dy_2. \end{aligned}$$

Moreover, the cuboid  $Q$  in Lemma 5.2 is defined by the conditions

$$|u_1| \leq 2^{-M}, \quad |u_2| \leq 2^{-\frac{M}{3}}, \quad |u_3| \leq 1, \quad |u_4| \leq 1, \quad |u_5| \leq 1.$$

In order to estimate the  $C^1$ -norm of  $H$  on  $Q$ , observe first that here we may estimate

$$(5.24) \quad \left| \check{\chi}_1(y_1) \check{\chi}_1 \left( u_4 - u_5 y_2 - y_2^3 b(x_1 - \lambda_1^{-1} y_1, u_2 y_2, \delta) + u_1 r_2(y_1; \lambda_1^{-1}, x_1, \delta) \right. \right. \\ \left. \left. + (u_2 y_2) \delta_3 r_3(y_1; \lambda_1^{-1}, x_1, \delta) \right) \right| \\ \leq C_N (1 + |y_1|)^{-N} (1 + |y_2|)^{-N},$$

for every  $N \in \mathbb{N}$  (just distinguish the cases where  $|y_1| \ll |y_2|^3$ , and  $|y_1| \gtrsim |y_2|^3$ ). Notice that this estimate allows in particular to absorb any powers of  $y_1$  or  $y_2$  in the upcoming estimations. Moreover, we find that

$$(5.25) \quad |H(u)| \leq C_N \iint (1 + |y_1|)^{-N} (1 + |y_2|)^{-N} |\check{\chi}_1(D - u_3 y_2 + r_1(y_1; \lambda_1^{-1}, x_1, \delta))| dy_1 dy_2.$$

It is easy to see that this allows to estimate

$$|H(u)| \leq C_N |D|^{-N/2}, \quad u \in Q.$$

Indeed, when  $1 + |y_1| \gtrsim |D|$ , then we can gain a factor  $|D|^{-N/2}$  from the first factor in the integral in (5.25), and when  $1 + |y_1| \ll |D|$  and  $|u_3 y_2| \leq |D|/2$ , then the last factor in the integral can be estimated by  $C'_N |D|^{-N}$ . Finally, when  $1 + |y_1| \ll |D|$  and  $|u_3 y_2| \geq |D|/2$ , then  $|y_2| \geq |D|/2$ , because  $|u_3| \leq 1$ , and we can gain a factor  $|D|^{-N/2}$  from the second factor in the integral in (5.25).

Similar estimates hold true also for partial derivatives of  $H$ , since these essentially produce only further factors of the order  $|y_2|, |r_2(y_1)| \lesssim (1 + |y_1|)$  and  $|y_2 r_3(y_1)| \lesssim |y_2|(1 + |y_1|)$  under the integral defining  $H(u)$ , and as we have observed before, such factors can easily be absorbed.

We thus find that  $\|H\|_{C^1(Q)} \lesssim |D|^{-1}$ , so that, by Lemma 5.2,

$$|\sigma^3(\lambda_1, t, x)| \lesssim \frac{1}{|2^{-it} - 1|} \left( \lambda_1 |Q_D(x)| \right)^{-1}.$$

This estimate allows to sum in  $\lambda_1$ , since we are assuming that  $\lambda_1 |Q_D(x)| \geq 4c$  in the definition of  $\mu_{1+it}^3(x)$ , and we conclude that also  $|\mu_{1+it}^2(x)| \leq C$ , provided we choose the second factor in the definition of  $\gamma(\zeta)$  as  $\gamma_2(\zeta) := (2^{1-\zeta} - 1)/(2^{2/3} - 1)$ .

**4. The part where  $\max\{|A|, |B|\} < 1$  and  $|D| < 4c$ .** Denote by  $\mu_{1+it}^4(x)$  the contribution to  $\mu_{1+it}(x)$  by the terms for which  $\max\{|A(x, \lambda_3, \delta)|, |B(x, \lambda_3, \delta)|\} < 1$  and  $|D(x, \lambda_1, \delta)| < 4c$ .

Under the assumptions of this case, it is easily seen from formula (5.18), in combination with an estimate analogous to (5.25), that

$$\mu_{\lambda_1, \lambda_3}(x) = J(A, B, D, E, \lambda_1^{-1}, \lambda_3^{-\frac{1}{3}}, \lambda_3 \lambda_1^{-1}),$$

where  $J$  is a smooth function of all its (bounded) variables. We may thus invoke Lemma 8.1 from [21] on oscillatory double sums in order to conclude that also  $|\mu_{1+it}^4(x)| \leq C$ , provided we choose the third factor  $\gamma_3(\zeta)$  of  $\gamma(\zeta)$  according to Remark 8.2 in [21].

Since the details are very similar to the discussion the corresponding case in the last part of the proof of Proposition 5.2 in [21], we shall skip the details.

Estimate (5.15) is a consequence of our estimates on the  $\mu_{1+it}^j(x), j = 1, \dots, 4$ , which completes the proof of Proposition 5.3.

Q.E.D.

**5.3. Estimation of  $T_\delta^{III_0}$ : Complex interpolation.** The discussion of this operator is easier than the one in the preceding subsection. Observe that in place of (5.9), we here have

$$(5.26) \quad \phi_\delta(x) := x_2^2 b(x_1, x_2, \delta) + x_1^n \alpha(\delta_1 x_1) + \delta_3 x_2 x_1^{n_1} \alpha_1(\delta_1 x_1), \quad (x_1 \sim 1, |x_2| < \varepsilon),$$

since  $B = 2$ .

Assuming again without loss of generality that  $\lambda_1 = \lambda_2$ , we see that we have to prove

**Proposition 5.4.** *Let  $m = 2$  and  $B = 2$ , and consider the measure*

$$\nu_\delta^{III_0} := \sum_{2^M \leq \lambda_3 \leq 2^{-M} \delta_0^{-1}} \sum_{2^M \lambda_3 \leq \lambda_1 \leq 2^{-M} \delta_0^{-1}} \nu_\delta^{(\lambda_1, \lambda_1, \lambda_3)},$$

where summation is taken over all sufficiently large dyadic  $\lambda_i$  in the given range. If we denote by  $T_\delta^{III_0}$  the operator of convolution with  $\widehat{\nu_\delta^{III_0}}$ , then, if  $M \in \mathbb{N}$  is sufficiently large (and  $\varepsilon$  sufficiently small),

$$(5.27) \quad \|T_\delta^{III_0}\|_{6/5 \rightarrow 6} \leq C,$$

with a constant  $C$  not depending on  $\delta$ , for  $\delta$  sufficiently small.

*Proof.* For fixed  $\lambda_3$  satisfying  $2^M \leq \lambda_3 \leq 2^{-M} \delta_0^{-1}$  we put

$$\sigma^{\lambda_3} := \sum_{\{\lambda_1: 2^M \lambda_3 \leq \lambda_1 \leq 2^{-M} \delta_0^{-1}\}} \nu_\delta^{(\lambda_1, \lambda_1, \lambda_3)},$$

so that

$$\nu_\delta^{III_0} = \sum_{2^M \leq \lambda_3 \leq 2^{-M} \delta_0^{-1}} \sigma^{\lambda_3}.$$

We embed  $\sigma^{\lambda_3}$  into an analytic family of measures

$$\sigma_\zeta^{\lambda_3}(x) := \gamma(\zeta) \sum_{\{\lambda_1: 2^M \lambda_3 \leq \lambda_1 \leq 2^{-M} \delta_0^{-1}\}} \lambda_1^{\frac{1-3\zeta}{2}} \nu_\delta^{(\lambda_1, \lambda_1, \lambda_3)}, \quad \zeta \in \Sigma,$$

where  $\gamma(\zeta) := 2^{3(1-\zeta)/2} - 1$ , so that  $\sigma_{1/3}^{\lambda_3} = \sigma^{\lambda_3}$ . From (4.15) we obtain that

$$\|\widehat{\sigma_{it}^{\lambda_3}}\|_\infty \leq C \lambda_3^{-\frac{1}{2}} \quad \forall t \in \mathbb{R}.$$

We shall also prove that for every sufficiently small  $\epsilon > 0$  there is a constant  $C_\epsilon$  such that

$$\|\sigma_{1+it}^{\lambda_3}\|_\infty \leq C_\epsilon \lambda_3^{\frac{1+\epsilon}{2}} \quad \forall t \in \mathbb{R}.$$

By Stein's interpolation theorem for analytic families of operators, these estimates easily imply that

$$\|T^{\lambda_3}\|_{p_c \rightarrow p'_c} \lesssim (\lambda_3^{-\frac{1}{2}})^{\frac{2}{3}} (\lambda_3^{\frac{1+\epsilon}{2}})^{\frac{1}{3}} = \lambda_3^{\frac{\epsilon-1}{6}},$$

where  $T^{\lambda_3}$  denotes the operator of convolution with  $\widehat{\sigma^{\lambda_3}}$ . Thus, if we choose  $\epsilon$  sufficiently small, we can also sum in  $\lambda_3$  and obtain (5.27).

Our goal will thus be to show that for  $\epsilon$  sufficiently small, we have

$$(5.28) \quad |\sigma_{1+it}^{\lambda_3}(x)| \leq C_\epsilon \lambda_3^{\frac{1+\epsilon}{2}},$$

where  $C_\epsilon$  is independent of  $t, x, \delta$  and  $\lambda_3$ .

To this end, observe that

$$\sigma_{1+it}^{\lambda_3}(x) := \gamma(1+it) \sum_{\{\lambda_1: 2^M \lambda_3 \leq \lambda_1 \leq 2^{-M} \delta_0^{-1}\}} \lambda_1^{-\frac{3}{2}it} \lambda_1^{-1} \nu_\delta^{(\lambda_1, \lambda_1, \lambda_3)}(x),$$

and, by (4.1), (4.2),

$$(5.29) \quad \begin{aligned} \lambda_1^{-1} \nu_\delta^{(\lambda_1, \lambda_1, \lambda_3)}(x) &= \lambda_1 \lambda_3 \int \tilde{\chi}_1\left(\lambda_1(x_1 - y_1)\right) \tilde{\chi}_1\left(\lambda_1(x_2 - \delta_0 y_2 - y_1^2 \omega(\delta_1 y_1))\right) \\ &\quad \tilde{\chi}_1\left(\lambda_3\left(x_3 - y_2^2 b(y_1, y_2, \delta) - y_1^n \alpha(\delta_1 y_1) - \delta_3 y_2 y_1^{n_1} \alpha_1(\delta_1 y_1)\right)\right) \eta(y) dy, \end{aligned}$$

where  $\eta$  is supported where  $y_1 \sim 1$  and  $|y_2| < \varepsilon$ . Now, if  $|x_1| \gg 1$ , or  $|x_2| \gg 1$ , then similar arguments as in the preceding subsection show that  $|\lambda_1^{-1} \nu_\delta^{(\lambda_1, \lambda_1, \lambda_3)}(x)| \lesssim \lambda_1^{-N} \lambda_3 \leq \lambda_3^{-1} \lambda_1^{2-N}$  for every  $N \geq 2$ , which implies (5.28).

We shall therefore assume from now on that  $|x_1| + |x_2| \lesssim 1$ . The change of variables  $y_1 \mapsto x_1 - y_1/\lambda_1$ ,  $y_2 \mapsto y_2/\lambda_3^{1/2}$  then leads in a similar way as in the previous subsection to  $\lambda_1^{-1} \nu_\delta^{(\lambda_1, \lambda_1, \lambda_3)}(x) = \lambda_3^{1/2} \mu_{\lambda_1, \lambda_3}(x)$ , where

$$\mu_{\lambda_1, \lambda_3}(x) := \iint \tilde{\chi}_1(y_1) F_\delta(\lambda_1, \lambda_3, x, y_1, y_2) dy_1 dy_2,$$

with

$$\begin{aligned} F_\delta(\lambda_1, \lambda_3, x, y_1, y_2) &:= \eta(x_1 - \lambda_1^{-1} y_1, \lambda_3^{-\frac{1}{2}} y_2) \tilde{\chi}_1(D - E y_2 + r_1(y_1)) \\ &\times \tilde{\chi}_1\left(A - B y_2 - y_2^2 b(x_1 - \lambda_3^{-1} (\lambda_3 \lambda_1^{-1} y_1), \lambda_3^{-\frac{1}{2}} y_2, \delta) + \lambda_3 \lambda_1^{-1} \left(r_2(y_1) + (\lambda_3^{-\frac{1}{2}} y_2) \delta_3 r_3(y_1)\right)\right). \end{aligned}$$

Here, the quantities  $A$  to  $E$  are given by

$$\begin{aligned} A &:= \lambda_3 Q_A(x), & B &:= \lambda_3^{\frac{1}{2}} Q_B(x), \\ D &:= \lambda_1 Q_D(x), & E &:= (\delta_0 \lambda_1) \lambda_3^{-\frac{1}{2}}, \end{aligned}$$

with  $Q_A(x)$ ,  $Q_B(x)$  and  $Q_D(x)$  as in (5.19). The functions  $r_i(y_1)$  have properties as before, and we choose again  $c > 0$  so that  $|r_i(y_1)| \leq c(1 + |y_1|)$ ,  $i = 1, 2, 3$ . Observe also that in this integral,  $|y_1| \lesssim \lambda_1$  and  $|y_2| \ll \lambda_3^{1/2}$ , and that only  $D$  and  $E$  depend on the summation variable  $\lambda_1$ .

It will be useful to observe that a simple van der Corput estimate allows to show that

$$(5.30) \quad \int \left| \chi_1 \left( A - By_2 - y_2^2 b(x_1 \dots, \delta) + \lambda_3 \lambda_1^{-1} \left( r_2(y_1) + (\lambda_3^{-\frac{1}{2}} y_2) \delta_3 r_3(y_1) \right) \right) \right| dy_2 \leq C,$$

with a constant  $C$  which does not depend on  $A, B, x, y_1$ , the  $\lambda_j$  and  $\delta$ .

**1. The part where  $|D| \geq 4c$ .** Denote by  $\sigma_{1+it,1}^{\lambda_3}(x)$  the contribution to  $\sigma_{1+it}^{\lambda_3}(x)$  by the terms for which  $|D(x, \lambda_1, \delta)| \geq 4c$ . We claim that, for every  $N \in \mathbb{N}$ ,

$$(5.31) \quad |\mu_{\lambda_1, \lambda_3}(x)| \lesssim |D|^{-N}.$$

Clearly, this estimate will allow us to sum in  $\lambda_1$  and obtain the right kind of estimate  $|\sigma_{1+it,1}^{\lambda_3}(x)| \leq C \lambda_3^{1/2}$ , in agreement with (5.28).

In order to prove (5.31), let us first consider the contribution  $\mu_{\lambda_1, \lambda_3}^1(x)$  to the integral defining  $\mu_{\lambda_1, \lambda_3}(x)$  by the region where  $|y_1| > |D|/4c$ . Here we may estimate  $|\check{\chi}_1(y_1)| \lesssim |D|^{-N} (1 + |y_1|)^{-N}$  for every  $N \in \mathbb{N}$ , and combining this with (5.30) clearly yields (5.31).

Denote next by  $\mu_{\lambda_1, \lambda_3}^2(x)$  the contribution by the region where  $|y_1| \leq |D|/4c$ . Then  $|r_i(y_1)| \leq |D|/2$ . But notice also that

$$|Ey_2| \ll |E| \lambda_3^{\frac{1}{2}} \leq \lambda_1 \delta_0 \ll 1,$$

which shows that  $|D - Ey_2 + r_1(y_1)| \geq |D|/4$ . Thus, the second factor in  $F_\delta$  can be estimated by  $C_N |D|^{-N}$ , and again we arrive at (5.31).

**2. The part where  $|D| < 4c$ .** Denote by  $\sigma_{1+it,2}^{\lambda_3}(x)$  the contribution to  $\sigma_{1+it}^{\lambda_3}(x)$  by the terms for which  $|D(x, \lambda_1, \delta)| < 4c$ .

As in the discussion in the previous subsection (part 2) we see that the oscillatory sum defining  $\lambda_3^{-1/2} \sigma_{1+it,2}^{\lambda_3}(x)$  can essentially be written in the form (5.11), with  $\alpha := -3/2$ ,  $l = k_1$  and

$$u_1 = 2^{\beta^{1l}} a_1 := \lambda_1^{-1} \lambda_3, \quad u_2 = 2^{\beta^{2l}} a_2 := \lambda_1 Q_D(x), \quad u_3 = 2^{\beta^{3l}} a_3 := \lambda_1 (\lambda_3^{-\frac{1}{2}} \delta_0)$$

and where the function  $H = H_{\lambda_3, x, \delta}$  of  $u := (u_1, u_2, u_3)$  is now given by

$$\begin{aligned} H(u) := & \iint \check{\chi}_1(y_1) \eta(x_1 - u_1 \lambda_3^{-1} y_1, \lambda_3^{-\frac{1}{2}} y_2) \check{\chi}_1(u_2 - u_3 y_2 + r_1(y_1; \lambda_3^{-1} u_1, x_1, \delta)) \\ & \times \check{\chi}_1 \left( A - By_2 - y_2^2 b(x_1 - \lambda_3^{-1} u_1 y_1, \lambda_3^{-\frac{1}{2}} y_2, \delta) + u_1 r_2(y_1; \lambda_3^{-1} u_1, x_1, \delta) \right. \\ & \left. + (\lambda_3^{-\frac{1}{2}} y_2) \delta_3 r_3(y_1; \lambda_3^{-1} u_1, x_1, \delta) \right) dy_1 dy_2. \end{aligned}$$



Moreover, the cuboid  $Q$  in Lemma 5.2 is defined by the conditions

$$|u_1| \leq 2^{-M}, \quad |u_2| < 4c, \quad |u_3| \leq 2^{-\frac{3}{2}M}.$$

Let us estimate the  $C^1$ - norm of  $H$  on  $Q$ . Because of (5.30), we clearly have  $\|H\|_{C(Q)} \lesssim 1$ . We next consider partial derivatives of  $H$ . From our integral formula for  $H(u)$ , it is obvious that the partial derivative of  $H$  with respect to  $u_1$  will essentially only produce additional factors of the form  $\lambda_3^{-1}y_1, \lambda_3^{-1}y_1y_2^2, \lambda_3^{-1/2}y_2\lambda_3^{-1}y_1$  under the double integral. However, powers of  $y_1$  can be absorbed by the rapidly decaying factor  $\tilde{\chi}_1(y_1)$ , and  $|\lambda_3^{-1}y_2^2| \leq \varepsilon \ll 1$ , so that  $|\partial_{u_1}H(u)| \lesssim 1$  too, and the same applies to  $|\partial_{u_2}H(u)|$ . The main problem is again caused by the partial derivative with respect to  $u_3$ , which produces an additional factor  $y_2$ .

However, arguing as in the preceding subsection (compare (5.15)), we find that for  $\epsilon \in ]0, 1]$  and  $s \in [0, 1]$

$$\begin{aligned} |\partial_{u_3}H(su)| &\lesssim |u_3|^{\epsilon-1} s^{\epsilon-1} \iint_{|y_2| \leq \lambda_3^{\frac{1}{2}}} (1 + |y_1|)^{-N} \left| \tilde{\chi}_1(A - By_2 - y_2^2b(x_1 - \lambda_3^{-1}su_1y_1, \lambda_3^{-\frac{1}{2}}y_2, \delta) \right. \\ &\quad \left. + su_1r_2(y_1; \lambda_3^{-1}su_1, x_1, \delta) + (\lambda_3^{-\frac{1}{2}}y_2)\delta_3r_3(y_1; \lambda_3^{-1}su_1, x_1, \delta) \right| |y_2|^\epsilon dy_1 dy_2. \end{aligned}$$

Estimating  $|y_2|^\epsilon$  in a trivial way by  $|y_2|^\epsilon \leq \lambda_3^{\epsilon/2}$ , we see by means of (5.30) that

$$(5.32) \quad |\partial_{u_3}H(su)| \lesssim |u_3|^{\epsilon-1} s^{\epsilon-1} \lambda_3^{\frac{\epsilon}{2}}, \quad \text{for all } u \in Q.$$

By means of Lemma 5.2 (and our choice of  $\gamma(\zeta)$ ), this implies that

$$|\lambda_3^{-1/2}\sigma_{1+it,2}^{\lambda_3}(x)| \lesssim \lambda_3^{\frac{\epsilon}{2}},$$

which completes the proof of (5.28), and hence also of Proposition 5.4.

Q.E.D.

**5.4. Estimation of  $T_\delta^{III_2}$ : Complex interpolation.** The discussion of this operator will somewhat resemble the one of the operator  $T_{\delta,j}^{VI}$  in Subsection 8.2 of [21], which arose from the same Subcase 3.2 (b) of Subsection 5.3 in [21], with  $2^{-j}$  playing again the role of  $\delta_0$  here, and where we have had  $B = 2$ , in place of  $B = 3$  here.

Assuming again without loss of generality that  $\lambda_1 = \lambda_2$ , we see that here we have to prove

**Proposition 5.5.** *Let  $m = 2$  and  $B = 3$ , and consider the measure*

$$\nu_\delta^{III_2} := \sum_{2^M \delta_0^{-1} \leq \lambda_1 < \delta_0^{-3/2}} \sum_{2^M \lambda_1 \delta_0 \leq \lambda_3 \leq 2^{-M}(\lambda_1 \delta_0)^3} \nu_\delta^{(\lambda_1, \lambda_1, \lambda_3)},$$

where summation is taken over all sufficiently large dyadic  $\lambda_i$  in the given range. If we denote by  $T_\delta^{III_2}$  the operator of convolution with  $\widehat{\nu_\delta^{III_2}}$ , then, if  $M \in \mathbb{N}$  is sufficiently large (and  $\varepsilon$  sufficiently small),

$$(5.33) \quad \|T_\delta^{III_2}\|_{6/5 \rightarrow 6} \leq C,$$

with a constant  $C$  not depending on  $\delta$ , for  $\delta$  sufficiently small.

*Proof.* Recall that, by (4.15),  $\|\widehat{\nu_\delta^\lambda}\|_\infty \lesssim \lambda_1^{-1/2} \lambda_3^{-1/3}$ . In analogy to the proof of Proposition 5.3, we therefore define here for  $\zeta$  in the strip  $\Sigma = \{\zeta \in \mathbb{C} : 0 \leq \operatorname{Re} \zeta \leq 1\}$  an analytic family of measures by

$$\mu_\zeta(x) := \gamma(\zeta) \sum_{2^M \delta_0^{-1} \leq 2^{k_1} < \delta_0^{-3/2}} \sum_{2^{M+k_1} \delta_0 \leq 2^{k_3} \leq 2^{-M} 2^{3k_1} \delta_0^3} 2^{\frac{(1-3\zeta)k_1}{2}} 2^{\frac{(1-3\zeta)k_3}{3}} \nu_\delta^{(2^{k_1}, 2^{k_1}, 2^{k_3})},$$

where we may here put

$$\gamma(\zeta) := \frac{1 - 2^{\frac{9}{2}(1-\zeta)}}{1 - 2^3}.$$

By  $T_\zeta$  we denote again the operator of convolution with  $\widehat{\mu_\zeta}$ . Observe that for  $\zeta = \theta_c = 1/3$ , we have  $\mu_{\theta_c} = \nu_\delta^{III_2}$ , hence  $T_{\theta_c} = T_\delta^{III_2}$ , so that, arguing exactly as in the preceding subsection, by means of Stein's interpolation theorem (5.33) will follow if we can prove that there is a constant  $C$  such that

$$(5.34) \quad |\mu_{1+it}(x)| \leq C,$$

where  $C$  is independent of  $t, x$  and  $\delta$ .

Setting

$$\mu_{\lambda_1, \lambda_3}(x) := \lambda_1^{-1} \lambda_3^{-\frac{2}{3}} \nu_\delta^{(\lambda_1, \lambda_1, \lambda_3)}(x),$$

we may re-write

$$(5.35) \quad \mu_{1+it}(x) = \gamma(1+it) \sum_{2^M \delta_0^{-1} \leq \lambda_1 < \delta_0^{-3/2}} \sum_{2^M \lambda_1 \delta_0 \leq \lambda_3 \leq 2^{-M} (\lambda_1 \delta_0)^3} \lambda_1^{-\frac{3}{2}it} \lambda_3^{-it} \mu_{\lambda_1, \lambda_3}(x).$$

Arguing as in the preceding subsection, by means the identity (5.17) we see again that we may assume in the sequel that  $|x_1| \sim 1$  and  $|x_2| \lesssim 1$  (notice that also here we have  $\lambda_3 \ll \lambda_1$ ).

By means of the change of variables  $y_1 \mapsto x_1 - y_1/\lambda_1$ ,  $y_2 \mapsto y_2/(\lambda_1 \delta_0)$  and Taylor expansion around  $x_1$  we may re-write  $\mu_{\lambda_1, \lambda_3}(x) = \lambda_3^{1/3}/(\lambda_1 \delta_0) \tilde{\mu}_{\lambda_1, \lambda_3}(x)$ , where

$$(5.36) \quad \tilde{\mu}_{\lambda_1, \lambda_3}(x) = \iint \tilde{\chi}_1(y_1) \tilde{F}_\delta(\lambda_1, \lambda_3, x, y_1, y_2) dy_1 dy_2,$$

with

$$\begin{aligned} \tilde{F}_\delta(\lambda_1, \lambda_3, x, y_1, y_2) &:= \eta(x_1 - \lambda_1^{-1} y_1, \delta_0^{-1} \lambda_1^{-1} y_2) \tilde{\chi}_1(D - y_2 + r_1(y_1)) \\ &\times \tilde{\chi}_1\left(A - B y_2 - E y_2^3 b(x_1 - \lambda_1^{-1} y_1, \delta_0^{-1} \lambda_1^{-1} y_2, \delta) + \lambda_3 \lambda_1^{-1} \left(r_2(y_1) + (\delta_0^{-1} \lambda_1^{-1} y_2) \delta_3 r_3(y_1)\right)\right). \end{aligned}$$

Here, the quantities  $A$  to  $E$  are given by

$$(5.37) \quad \begin{aligned} A &= A(x, \lambda_3, \delta) := \lambda_3 Q_A(x), & B &:= B(x, \lambda_1, \lambda_3, \delta) := \frac{\lambda_3}{\lambda_1} Q_B(x), \\ D &= D(x, \lambda_1, \delta) := \lambda_1 Q_D(x), & E &:= \frac{\lambda_3}{(\delta_0 \lambda_1)^3} \leq 2^{-M}, \end{aligned}$$

with

$$Q_A(x) := x_3 - x_1^n \alpha(\delta_1 x_1), \quad Q_B(x) := \frac{\delta_3}{\delta_0} x_1^{n_1} \alpha_1(\delta_1 x_1), \quad Q_D(x) := x_2 - x_1^2 \omega(\delta_1 x_1).$$

Again, the functions  $r_i(y_1) = r_i(y_1; \lambda_1^{-1}, x_1, \delta)$ ,  $i = 1, 2, 3$ , are smooth functions of  $y_1$  (and  $\lambda_1^{-1}$  and  $x_1$ ) satisfying estimates of the form (5.20). Moreover, we may assume that  $|y_1| \lesssim \lambda_1$  and  $|\delta_0^{-1} \lambda_1^{-1} y_2| \leq \varepsilon$ , because of our assumption  $|x_1| \sim 1$  and the support properties of  $\eta$ .

The factor  $\lambda_3^{1/3}/(\lambda_1 \delta_0)$  by which  $\mu_{\lambda_1, \lambda_3}(x)$  and  $\tilde{\mu}_{\lambda_1, \lambda_3}(x)$  differ suggests to decompose the summation over  $k_3$  into three arithmetic progressions  $k_3 = i + 3k_4$ ,  $i = 0, 1, 2$  (cf. a similar discussion in [21]). Restricting ourselves to anyone of them, let us assume for simplicity that  $i = 0$ , so that  $k_3 = 3k_4$ , with  $k_4 \in \mathbb{N}$ . Let us also assume that  $\delta_0$  is a dyadic number (otherwise, replace  $\delta_0$  by the biggest dyadic number smaller or equal to  $\delta_0$ ). It is then convenient to introduce new summation variables  $(k_0, k_4)$  in place of  $(k_1, k_3)$  by requiring that  $k_1 = k_0 + k_3/3 - \log_2(\delta_0) = k_0 + k_4 - \log_2(\delta_0)$ . In terms of their exponentials  $\lambda_0 := 2^{k_0}$  and  $\lambda_4 := 2^{k_4}$ , this means that

$$\lambda_1 = \frac{\lambda_0 \lambda_4}{\delta_0}, \quad \lambda_3 = \lambda_4^3,$$

and we can re-write the conditions on the index sets for  $\lambda_1$  and  $\lambda_3$  over which we sum in (5.35) as

$$(5.38) \quad \lambda_0 \geq 2^{\frac{M}{3}}, \quad 2^{\frac{M}{2}} \lambda_0^{\frac{1}{2}} \leq \lambda_4 \leq \delta_0^{-\frac{1}{2}} \lambda_0^{-1},$$

and correspondingly we shall re-write (5.35) as

$$\mu_{1+it}(x) = \gamma(1+it) \delta_0^{\frac{3}{2}it} \sum_{\lambda_0 \geq 2^{\frac{M}{3}}} \sum_{2^{\frac{M}{2}} \lambda_0^{\frac{1}{2}} \leq \lambda_4 \leq \delta_0^{-\frac{1}{2}} \lambda_0^{-1}} \lambda_0^{-(1+\frac{3}{2}it)} \lambda_4^{-\frac{9}{2}it} \tilde{\mu}_{\frac{\lambda_0 \lambda_4}{\delta_0}, \lambda_4^3}(x).$$

For  $\lambda_0$  and  $x$  fixed, let us put

$$\begin{aligned} f_{\lambda_0, x}(\lambda_4) &:= \tilde{\mu}_{\frac{\lambda_0 \lambda_4}{\delta_0}, \lambda_4^3}(x), \\ \rho_{t, \lambda_0}(x) &:= \gamma(1+it) \sum_{\{\lambda_4: 2^{\frac{M}{2}} \lambda_0^{\frac{1}{2}} \leq \lambda_4 \leq \delta_0^{-\frac{1}{2}} \lambda_0^{-1}\}} \lambda_4^{-\frac{9}{2}it} f_{\lambda_0, x}(\lambda_4). \end{aligned}$$

The previous formula for  $\mu_{1+it}(x)$  shows that in order to verify (5.34), it will suffice to prove the following uniform estimate: there exist constants  $C > 0$  and  $\epsilon \geq 0$  with  $\epsilon < 1$ , so that for all  $x$  such that  $|x_1| + |x_2| \lesssim 1$  and  $\delta$  sufficiently small we have

$$(5.39) \quad |\rho_{t, \lambda_0}(x)| \leq C \lambda_0^\epsilon \quad \text{for } \lambda_0 \geq 2^{\frac{M}{3}}.$$

In order to prove this, observe that by (5.36)

$$(5.40) \quad f_{\lambda_0, x}(\lambda_4) = \iint \check{\chi}_1(y_1) F_\delta(\lambda_0, \lambda_4, x, y_1, y_2) dy_1 dy_2,$$

where

$$\begin{aligned} F_\delta(\lambda_0, \lambda_4, x, y_1, y_2) &:= \eta(x_1 - \delta_0(\lambda_0 \lambda_4)^{-1} y_1, (\lambda_0 \lambda_4)^{-1} y_2, \delta) \check{\chi}_1(D - y_2 + r_1(y_1)) \\ &\quad \times \check{\chi}_1 \left( A - B y_2 - E y_2^3 b(x_1 - \delta_0(\lambda_0 \lambda_4)^{-1} y_1, (\lambda_0 \lambda_4)^{-1} y_2, \delta) + \delta_3 \delta_0 \lambda_4 \lambda_0^{-2} y_2 r_3(y_1) \right. \\ &\quad \left. + \delta_0 \lambda_4^2 \lambda_0^{-1} r_2(y_1) \right) \end{aligned}$$

and

$$(5.41) \quad \begin{aligned} A &= A(x, \lambda_4, \delta) := \lambda_4^3 Q_A(x), & B &:= B(x, \lambda_0, \lambda_4, \delta) := \frac{\lambda_4^2}{\lambda_0} Q_B(x), \\ D &= D(x, \lambda_0, \lambda_4, \delta) := \frac{\lambda_0 \lambda_4}{\delta_0} Q_D(x), & E &:= \lambda_0^{-3} \leq 2^{-M}, \end{aligned}$$

where

$$Q_A(x) := x_3 - x_1^n \alpha(\delta_1 x_1), \quad Q_B(x) := \delta_3 x_1^{n_1} \alpha_1(\delta_1 x_1), \quad Q_D(x) := x_2 - x_1^2 \omega(\delta_1 x_1).$$

The functions  $r_i(y_1) = r_i(y_1; \lambda_0^{-1}, \lambda_4^{-1}, x_1, \delta)$ ,  $i = 1, 2, 3$ , are smooth functions of  $y_1$  (and  $\lambda_1^{-1}, \lambda_4^{-1}$  and  $x_1$ ), satisfying estimates of the form

$$(5.42) \quad |r_i(y_1)| \leq C|y_1|, \quad \left| \left( \frac{\partial}{\partial(\lambda_4^{-1})} \right) r_i(y_1; \lambda_0^{-1}, \lambda_4^{-1}, x_1, \delta) \right| \leq C_l |y_1|^2$$

(compare (5.20)).

Given  $x$  and  $\lambda_0$ , we shall split the summation in  $\lambda_4$  into sub-intervals, according to the (relative) sizes of the quantities  $A, B$  and  $D$ , which are considered as functions of  $\lambda_4$ .

**1. The part where  $|D| \gg 1$ .** Denote by  $\rho_{t, \lambda_0}^1(x)$  the contribution to  $\rho_{t, \lambda_0}(x)$  by the terms for which  $|D| \gg 1$ .

We first consider the contribution to  $f_{\lambda_0, x}(\lambda_4)$  given by integrating in (5.40) over the region where  $|y_1| \gtrsim |D|^\varepsilon$  (where  $\varepsilon > 0$  is assumed to be sufficiently small). Here, the rapidly decaying first factor  $\check{\chi}_1(y_1)$  leads to an improved estimate of this contribution of the order  $|D|^{-N}$  for every  $N \in \mathbb{N}$ , which allows to sum over the dyadic  $\lambda_4$  for which  $|D| \gg 1$ , and the contribution to  $\rho_{t, \lambda_0}^1(x)$  is of order  $O(1)$ , which is stronger than what is needed in (5.39).

We may therefore restrict ourselves in the sequel to the region where  $|y_1| \ll |D|^\varepsilon$ . Observe that, because of (5.42), this implies in particular that  $|r_i(y_1)| \ll |D|^\varepsilon$ ,  $i = 1, 2, 3$ . By looking at the second factor in  $F_\delta$ , we again see that the contribution by the regions where in addition  $|y_2| < |D|/2$ , or  $|y_2| > 3|D|/2$ , is again of the order  $|D|^{-N}$  for every  $N \in \mathbb{N}$ , and their contributions to  $\rho_{t, \lambda_0}^1(x)$  are again admissible.

What remains is the region where  $|y_1| \ll |D|^\varepsilon$  and  $|D|/2 \leq |y_2| \leq 3|D|/2$ . In addition, we may assume that  $y_2$  and  $D$  have the same sign, since otherwise we can estimate as before. Let us therefore assume, e.g., that  $D > 0$ , and that  $D/2 \leq y_2 \leq 3D/2$ .

The change of variables  $y_2 \mapsto Dy_2$  then allows to re-write the corresponding contribution to  $f_{\lambda_0, x}(\lambda_4)$  as

$$(5.43) \quad \tilde{f}_{\lambda_0, x}(\lambda_4) := D \int_{|y_1| \ll |D|^\varepsilon} \int_{1/2 \leq y_2 \leq 3/2} \check{\chi}_1(y_1) \tilde{F}_\delta(\lambda_0, \lambda_4, x, y_1, y_2) dy_2 dy_1,$$

where here

$$\begin{aligned} \tilde{F}_\delta(\lambda_0, \lambda_4, x, y_1, y_2) &:= \eta(x_1 - \delta_0(\lambda_0 \lambda_4)^{-1} y_1, (\lambda_0 \lambda_4)^{-1} D y_2, \delta) \check{\chi}_1(D - D y_2 + r_1(y_1)) \chi_1(y_2) \\ &\check{\chi}_1 \left( A - B D y_2 - E D^3 y_2^3 b(x_1 - \delta_0(\lambda_0 \lambda_4)^{-1} y_1, (\lambda_0 \lambda_4)^{-1} D y_2, \delta) + \delta_3 \delta_0 \lambda_4 \lambda_0^{-2} D r_3(y_1) y_2 \right. \\ &\quad \left. + \delta_0 \lambda_4^2 \lambda_0^{-1} r_2(y_1) \right), \end{aligned}$$

and where  $\chi_1$  is supported where  $y_2 \sim 1$ . In the subsequent discussion, we may and shall assume that  $f_{\lambda_0, x}(\lambda_4)$  is replaced by  $\tilde{f}_{\lambda_0, x}(\lambda_4)$ .

Recall also from (5.10) that  $b(x_1, x_2, \delta) = b_3(\delta_1 x_1, \delta_2 x_2)$ , and that  $\delta_2 \ll \delta_0$ . The last estimate implies that

$$|\delta_2(\lambda_0 \lambda_4)^{-1} D| = \frac{\delta_2}{\delta_0} |Q_D(x)| \ll 1,$$

which shows that the second derivative of the argument of the last factor of our function  $\tilde{F}_\delta(\lambda_0, \lambda_4, x, y_1, y_2)$  with respect to  $y_2$  is comparable to  $|E D^3|$ . We may therefore apply a classical van der Corput estimate for the integration in  $y_2$  (see [32]; also case (i) in Lemma 2.2 (b) in [21]) and obtain that

$$|\tilde{f}_{\lambda_0, x}(\lambda_4)| \lesssim |D| |E D^3|^{-\frac{1}{2}} = \lambda_0^{\frac{3}{2}} |D|^{-\frac{1}{2}}.$$

Interpolation with the trivial estimate  $|\tilde{f}_{\lambda_0, x}(\lambda_4)| \lesssim 1$  then leads to  $|\tilde{f}_{\lambda_0, x}(\lambda_4)| \lesssim \lambda_0^{\frac{1}{2}} |D|^{-\frac{1}{6}}$ . The second factor allows to sum in  $\lambda_4$ , since we are assuming that  $|D| \gg 1$ , and we obtain  $|\rho_{t, \lambda_0}^1(x)| \leq C \lambda_0^{1/2}$ , in agreement with (5.39).

**We may thus in the sequel assume that  $|D| \lesssim 1$ .** Here we go back to (5.40) and observe that  $\check{\chi}_1(y_1) \check{\chi}_1(D - y_2 + r_1(y_1))$  can be estimated by  $C_N (1 + |y_1|)^{-N} (1 + |y_2|)^{-N}$ . This shows in particular that any power of  $y_1$  or  $y_2$  can be “absorbed” by these two factors.

We shall still have to distinguish between the cases where  $|B| \geq 1$ , and where  $|B| < 1$ .

**2. The part where  $|D| \lesssim 1$  and  $|B| \geq 1$ .** Denote by  $\rho_{t, \lambda_0}^2(x)$  the contribution to  $\rho_{t, \lambda_0}(x)$  by the terms for which  $|D| \lesssim 1$  and  $|B| \geq 1$ .

If  $|y_2| \gtrsim (|B|/|E|)^{1/2}$ , then we see that we can estimate the contribution to  $f_{\lambda_0, x}(\lambda_4)$  by a constant times  $(|B|/|E|)^{-1/2} = \lambda_0^{-3/2} |B|^{-1/2}$ . Summing over all  $\lambda_4$  such that

$|B| \geq 1$  then leads to a uniform estimate for the contributions of these regions to  $\rho_{t,\lambda_0}^2(x)$ .

So, assume that  $|y_2| \ll (|B|/|E|)^{1/2} =: H$ . Applying the change of variables  $y_2 \mapsto Hy_2$ , we then see that we may replace  $f_{\lambda_0,x}(\lambda_4)$  by

$$(5.44) \quad \tilde{f}_{\lambda_0,x}(\lambda_4) = H \iint \tilde{\chi}_1(y_1) \tilde{F}_\delta(\lambda_0, \lambda_4, x, y_1, y_2) dy_1 dy_2,$$

where

$$\begin{aligned} \tilde{F}_\delta(\lambda_0, \lambda_4, x, y_1, y_2) &:= \eta(x_1 - \delta_0(\lambda_0\lambda_4)^{-1}y_1, (\lambda_0\lambda_4)^{-1}Hy_2, \delta) \tilde{\chi}_1(D - Hy_2 + r_1(y_1)) \\ &\quad \times \chi_0(y_2) \tilde{\chi}_1\left(A - HB\left(y_2 + y_2^3 \operatorname{sgn}(B) b(x_1 - \delta_0(\lambda_0\lambda_4)^{-1}y_1, (\lambda_0\lambda_4)^{-1}Hy_2, \delta)\right) \right. \\ &\quad \left. + \delta_3\delta_0\lambda_4\lambda_0^{-2}H y_2 r_3(y_1) + \delta_0\lambda_4^2\lambda_0^{-1}r_2(y_1)\right) \end{aligned}$$

We claim that

$$(5.45) \quad |\tilde{f}_{\lambda_0,x}(\lambda_4)| \leq CH|HB|^{-\frac{1}{2}} = \lambda_0^{\frac{3}{4}}|B|^{-\frac{1}{4}}.$$

Since we are here assuming that  $|B| \geq 1$ , this estimate would imply the estimate  $|\rho_{t,\lambda_0}^2(x)| \leq C\lambda_0^{3/4}$ , again in agreement with (5.39).

In order to prove (5.45), observe first that the contribution to  $\tilde{f}_{\lambda_0,x}(\lambda_4)$  by the region where  $|y_1| > |HB|$  clearly can be estimated by the right-hand side of (5.45), because of the rapidly decaying factor  $\tilde{\chi}_1(y_1)$  in the integrand. And, on the remaining region where  $|y_1| \leq |HB|$ , we have

$$\begin{aligned} |\delta_3\delta_0\lambda_4\lambda_0^{-2}H r_3(y_1)| &\leq \delta_3\delta_0\lambda_4\lambda_0^{-2}|H||HB| = \delta_3\frac{\delta_0\lambda_4^2}{\lambda_0}|Q_B(x)|^{\frac{1}{2}}|HB| \\ &\lesssim \lambda_0^{-3}\delta_3^{\frac{3}{2}}|HB| \ll |HB|. \end{aligned}$$

Observe also that  $(\lambda_0\lambda_4)^{-1}H = |Q_B(x)|^{\frac{1}{2}} \lesssim \delta_3^{\frac{1}{2}} \ll 1$ . This shows that if  $\gamma$  denotes the argument of the last factor of  $\tilde{F}_\delta(\lambda_0, \lambda_4, x, y_1, y_2)$ , then there are constants  $0 < C_1 < C_2$ , such that

$$C_1|HB| \leq \left|\frac{\partial}{\partial y_2}\gamma(y_2)\right| + \left|\left(\frac{\partial}{\partial y_2}\right)^2\gamma(y_2)\right| \leq C_2|HB|, \quad |y_2| \lesssim 1,$$

uniformly in  $x, y_1$  and  $\delta$ . We may thus apply a van der Corput type estimate (see case (ii) in Lemma 2.2 (b) of [21]) to the integration in  $y_2$  and again arrive at an estimate by the right-hand side of (5.45), also for the contribution by the region where  $|y_1| \leq |HB|$ .

**3. The part where  $|D| \lesssim 1$ ,  $|B| < 1$  and  $|A| \gg 1$ .** Denote by  $\rho_{t,\lambda_0}^3(x)$  the contribution to  $\rho_{t,\lambda_0}(x)$  by the terms for which these conditions are satisfied. We claim that here we get

$$(5.46) \quad |f_{\lambda_0,x}(\lambda_4)| \leq C|A|^{-N},$$

for every  $N \in \mathbb{N}$ . This estimate will imply the estimate  $|\rho_{t,\lambda_0}^3(x)| \leq C\lambda_0^{3/4}$ , again in agreement with (5.39).

In order to prove (5.46), observe that the contributions to  $f_{\lambda_0,x}(\lambda_4)$  by the regions where  $|y_1| \gtrsim |A|^{1/3}$ , or  $|y_2| \gtrsim |A|^{1/3}$ , can be estimated by a constant times  $(|A|^{1/3})^{-N}$ , because of the rapid decay in  $y_1$  and  $y_2$  of  $F_\delta$ . So, assume that  $|y_1| + |y_2| \ll |A|^{1/3}$ . Then we see that

$$|By_2| \ll |B||A|^{1/3} \ll |A| \quad \text{and} \quad |Ey_2^3| \ll |EA| \ll |A|,$$

as well as

$$|\delta_3\delta_0\lambda_4\lambda_0^{-2}y_2r_3(y_1) + \delta_0\lambda_4^2\lambda_0^{-1}r_2(y_1)| \ll |A|^{\frac{2}{3}} \ll |A|,$$

and thus the last factor of  $F_\delta$  is of order  $|A|^{-N}$ . Consequently, also the contribution to  $f_{\lambda_0,x}(\lambda_4)$  by the region where  $|y_1| + |y_2| \ll |A|^{1/3}$  can be estimated as in (5.46).

**4. The part where  $\max\{|A|, |B|, |D|\} \lesssim 1$ .** Denote by  $\rho_{t,\lambda_0}^4(x)$  the contribution to  $\rho_{t,\lambda_0}(x)$  by the terms for which  $\max\{|A|, |B|, |D|\} \lesssim 1$ . Then we can easily estimate  $\rho_{t,\lambda_0}^4(x)$  by means of Lemma 5.2 in a very similar way as we did in the last part of Section 8 in [21], and obtain that  $|\rho_{t,\lambda_0}^4(x)| \leq C$ .

This completes the proof of estimate (5.39) (with  $\epsilon := 3/4$ ), and hence also of the proof of Proposition 5.5.

Q.E.D.

## 6. THE CASE WHERE $\lambda_1 \sim \lambda_2 \sim \lambda_3$

We shall assume for the sake of simplicity that

$$\lambda_1 = \lambda_2 = \lambda_3 \gg 1.$$

The more general case where  $\lambda_1 \sim \lambda_2 \sim \lambda_3 \gg 1$  can be treated in a very similar way. By changing notation slightly, we shall denote in this section by  $\lambda$  the common value of the  $\lambda_j$ .

We change coordinates from  $\xi = (\xi_1, \xi_2, \xi_3)$  to  $s_1, s_2$  and  $s_3 := \xi_3/\lambda$ , i.e.,

$$\xi_1 = s_1\xi_3 = \lambda s_1 s_3, \quad \xi_2 = \lambda s_2 \xi_3 = \lambda s_2 s_3, \quad \xi_3 = \lambda s_3,$$

and write in the sequel

$$s := (s_1, s_2, s_3) \quad s' := (s_1, s_2).$$

Then we may re-write

$$\Phi(x, \delta, \xi) = \lambda s_3 \tilde{\Phi}(x, s'),$$

where

$$\begin{aligned} \tilde{\Phi}(x, s') &:= s_1 x_1 + s_2 x_1^m \omega(\delta_1 x_1) + x_1^n \alpha(\delta_1 x_1) \\ (6.1) \quad &+ s_2 \delta_0 x_2 + \left( x_2^B b(x_1, x_2, \delta) + r(x_1, x_2, \delta) \right), \end{aligned}$$

where  $\omega(0) \neq 0, \alpha(0) \neq 0$ , and  $b(x_1, 0, \delta) \neq 0$ , if  $x_1 \sim 1$ , and where  $\delta$  and  $r(x_1, x_2, \delta)$  are given by (3.11) and (3.10), respectively.

Now, the first part of  $\tilde{\Phi}$  has at worst an Airy type singularity with respect to  $x_1$ , and the derivate of order  $B$  with respect to  $x_2$  does not vanish, so that we obtain

$$(6.2) \quad \|\widehat{\nu_\delta^\lambda}\|_\infty \lesssim \lambda^{-\frac{1}{3}-\frac{1}{B}}$$

(indeed, by localizing near a given point  $x^0$  and looking at the corresponding Newton polyhedron of  $\tilde{\Phi}$  at this point, this follows more precisely from the main result in [20]). On the other hand, standard van der Corput type arguments (compare Lemma 2.2 in [21]) show that here

$$(6.3) \quad \|\nu_\delta^\lambda\|_\infty \lesssim \min\{\lambda^3 \lambda^{-1} \lambda^{-\frac{1}{B}}, \lambda^3 \lambda^{-1} (\lambda \delta_0)^{-1}\} = \lambda \min\{\lambda^{\frac{B-1}{B}}, \delta_0^{-1}\}.$$

We therefore distinguish the following cases:

**Case A:**  $\lambda \leq \delta_0^{-B/(B-1)}$ . Then  $\|\nu_\delta^\lambda\|_\infty \lesssim \lambda^{2-1/B}$ , and by interpolation we get

$$(6.4) \quad \|T_\delta^\lambda\|_{p_c \rightarrow p'_c} \lesssim \lambda^{-\frac{1}{3}-\frac{1}{B}+\frac{7}{3}\theta_c},$$

again with  $\theta_c = 2/p'_c$ .

**Case B:**  $\lambda > \delta_0^{-B/(B-1)}$ . Then  $\|\nu_\delta^\lambda\|_\infty \lesssim \lambda \delta_0^{-1}$ , and by interpolation we get

$$(6.5) \quad \|T_\delta^\lambda\|_{p_c \rightarrow p'_c} \lesssim \lambda^{-\frac{1}{3}-\frac{1}{B}+(\frac{4}{3}+\frac{1}{B})\theta_c} \delta_0^{-\theta_c}.$$

Observe that in both cases, the exponents of  $\lambda$  in these estimates become strictly smaller if we replace  $\theta$  by a strictly smaller numbers, and the one of  $\delta$  increases.

**6.1. The case where  $h^r + 1 > B$ .** We observe that then  $p'_c > 2B$ , and thus

$$\theta_c < 1/B.$$

This shows that

$$-\frac{1}{3} - \frac{1}{B} + \frac{7}{3}\theta_c < -\frac{1}{3} - \frac{1}{B} + \frac{7}{3}\frac{1}{B} = -\frac{B-4}{3B}.$$

Thus, if  $B \geq 4$ , then for  $\theta = \theta_c$ , the exponent of  $\lambda$  in (6.4) is strictly negative, so that in Case A we can sum the estimates for  $p = p_c$  over all  $\lambda \leq \delta_0^{-B/(B-1)}$  and obtain, for  $B \geq 4$ ,

$$(6.6) \quad \sum_{\lambda \leq \delta_0^{-B/(B-1)}} \|T_\delta^\lambda\|_{p_c \rightarrow p'_c} \lesssim 1.$$

Similarly, since

$$-\frac{1}{3} - \frac{1}{B} + (\frac{4}{3} + \frac{1}{B})\theta_c < -\frac{1}{3} - \frac{1}{B} + (\frac{4}{3} + \frac{1}{B})\frac{1}{B} = -\frac{B^2 - B - 3}{3B^2},$$

where  $B^2 - B - 3 > 0$  if  $B \geq 3$ , we see that (6.5) implies in Case B that

$$\sum_{\lambda > \delta_0^{-B/(B-1)}} \|T_\delta^\lambda\|_{p_c \rightarrow p'_c} \lesssim \delta_0^{\frac{B+3-7B\theta_c}{3(B-1)}} < \delta_0^{\frac{B-4}{3(B-1)}},$$



hence, for  $B \geq 4$ ,

$$(6.7) \quad \sum_{\lambda > \delta_0^{-B/(B-1)}} \|T_\delta^\lambda\|_{p_c \rightarrow p'_c} \lesssim 1, \quad (B \geq 4).$$

The case where  $B = 3$  requires more refined estimates, whereas the case  $B = 2$  is rather easy to handle, given our assumption (3.15).

**Assume first that  $B = 2$ .** Then, by (6.4), if  $\lambda \leq \delta_0^{-2}$ ,

$$\|T_\delta^\lambda\|_{p_c \rightarrow p'_c} \lesssim \lambda^{-\frac{1}{3} - \frac{1}{2} + \frac{7}{3}\theta_c} = \lambda^{\frac{14\theta_c - 5}{6}},$$

and since by our assumption (3.15)  $\theta_c \leq 1/3$ , the exponent of  $\lambda$  in this estimate is strictly negative, so that we can sum over  $\lambda$  and again obtain (6.6).

Similarly, by (6.5), if  $\lambda > \delta_0^{-2}$ ,

$$\|T_\delta^\lambda\|_{p_c \rightarrow p'_c} \lesssim \lambda^{-\frac{1}{3} - \frac{1}{2} + (\frac{4}{3} + \frac{1}{2})\theta_c} \delta_0^{-\theta_c} = \lambda^{\frac{11\theta_c - 5}{6}} \delta_0^{-\theta_c}.$$

But,  $11\theta_c - 5 \leq 11/3 - 5 < 0$ , and so we get

$$\sum_{\lambda > \delta_0^{-2}} \|T_\delta^\lambda\|_{p_c \rightarrow p'_c} \lesssim \delta_0^{\frac{5 - 14\theta_c}{3}} \leq 1,$$

so that (6.7) holds true also in this case.

**Assume next that  $B = 3$ .** Then in Case A, where  $\lambda \leq \delta_0^{-3/2}$ , we have by (6.4)

$$\|T_\delta^\lambda\|_{p_c \rightarrow p'_c} \lesssim \lambda^{\frac{7\theta_c - 2}{3}},$$

and thus, if  $7\theta_c - 2 < 0$ , then we can sum these estimates in  $\lambda \leq \delta_0^{-3/2}$  and obtain (6.7).

Let us therefore assume henceforth that  $\theta_c \geq 2/7$ . Observe that by Lemma 3.1 we have  $\theta_c < \tilde{\theta}_c$ , unless  $\tilde{h}^r = d$  and  $h^r + 1 \geq H$ , in which case we have  $\theta_c = \tilde{\theta}_c$  and  $\tilde{p}'_c = p'_c$ . Thus

$$\|T_\delta^\lambda\|_{p_c \rightarrow p'_c} \lesssim \lambda^{\frac{7\tilde{\theta}_c - 2}{3}},$$

with  $\tilde{\theta}_c > 2/7$ , unless  $\theta_c = \tilde{\theta}_c = 2/7$ ,  $\tilde{h}^r = d$  and  $h^r + 1 \geq H$ . Note that in the latter case,  $H = B = 3$ , and since  $\tilde{\theta}_c = 2/7$ , we find that  $m = 5$  and  $d = 5.2$ .

In this particular case, we only get a uniform estimates for  $\|T_\delta^\lambda\|_{p_c \rightarrow p'_c} \lesssim 1$ . However, here we have  $\|\nu_\delta^\lambda\|_\infty \lesssim \lambda^{5/3}$ , since we are in Case A, and  $\|\widehat{\nu_\delta^\lambda}\|_\infty \lesssim \lambda^{2/3}$ , whereas  $\theta_c = 2/7$ , and thus  $-(1 - \theta_c)2/3 + \theta_c 5/3 = 0$ . Moreover,  $\nu_\delta^\lambda = \nu_\delta * \phi_\lambda$ , where the Fourier transform of  $\phi_\lambda$  is given by  $\widehat{\phi_\lambda}(\xi) = \chi_1\left(\frac{\xi_1}{\lambda}\right)\chi_1\left(\frac{\xi_2}{\lambda}\right)\chi_1\left(\frac{\xi_3}{\lambda}\right)$ . This implies a uniform estimate of the  $L^1$ -norms of the  $\phi_\lambda$  for all dyadic  $\lambda$ . We may thus estimate the operator  $T^{IV_1}$  of convolution with the Fourier transform of the complex measure

$$\nu_\delta^{IV_1} := \sum_{\lambda \leq \delta_0^{-3/2}} \nu_\delta^\lambda$$

by means of the real-interpolation Proposition 5.1 in the same way as we estimated the operators  $T^{I_1}$  and  $T^{II_1}$  in Subsection 5.1, by adding the measure  $\nu_\delta^{IV_1}$  to the family of measures  $\mu^i, i \in I$ , from the second class in (5.8).

So, assume that  $\tilde{\theta}_c > 2/7$ . Then we find that

$$\sum_{\lambda \leq \delta_0^{-3/2}} \|T_\delta^\lambda\|_{p_c \rightarrow p'_c} \lesssim \delta_0^{\frac{2-7\tilde{\theta}_c}{2}}.$$

Let us next turn to Case B, where  $\lambda > \delta_0^{-3/2}$ . Then, by (6.5),

$$\|T_\delta^\lambda\|_{p_c \rightarrow p'_c} \lesssim \lambda^{\frac{5\theta_c-2}{3}} \delta_0^{-\theta_c}.$$

Since  $\theta_c \leq 1/3$ , we can sum in  $\lambda$  and obtain

$$\sum_{\lambda > \delta_0^{-3/2}} \|T_\delta^\lambda\|_{p_c \rightarrow p'_c} \lesssim \delta_0^{\frac{2-7\theta_c}{2}} \leq \delta_0^{\frac{2-7\tilde{\theta}_c}{2}}.$$

Combining these estimates, we obtain

$$(6.8) \quad \sum_{\lambda \gg 1} \|T_\delta^\lambda\|_{p_c \rightarrow p'_c} \lesssim \delta_0^{\frac{2-7\tilde{\theta}_c}{2}}.$$

Observe next that

$$\frac{2-7\tilde{\theta}_c}{2} = 1 - \frac{7}{\tilde{p}'_c} = \frac{2\tilde{h}^r - 5}{\tilde{p}'_c},$$

and recall that  $\delta_0 = 2^{-(\tilde{\kappa}_2 - m\tilde{\kappa}_1)}$ . In combination with the re-scaling estimate (3.14) this leads to

$$(6.9) \quad \begin{aligned} \left( \int |\hat{f}|^2 d\mu_{1,k} \right)^{\frac{1}{2}} &\lesssim 2^{-k \left( \frac{|\tilde{\kappa}|}{2} - \frac{\tilde{\kappa}_1(1+m)+1}{\tilde{p}'_c} + (\tilde{\kappa}_2 - m\tilde{\kappa}_1) \frac{2\tilde{h}^r - 5}{2\tilde{p}'_c} \right)} \|f\|_{L^{p_c}} \\ &\leq C 2^{-k \left( \frac{|\tilde{\kappa}|}{2} - \frac{\tilde{\kappa}_1(1+m)+1}{\tilde{p}'_c} + (\tilde{\kappa}_2 - m\tilde{\kappa}_1) \frac{2\tilde{h}^r - 5}{2\tilde{p}'_c} \right)} \|f\|_{L^{p_c}}. \end{aligned}$$

where  $\mu_{1,k}$  denotes the measure corresponding to the frequency domains that we are here considering, i.e.,  $\mu_{1,k}$  corresponds to the re-scaled measure

$$\nu_{1,\delta} := \sum_{\lambda \gg 1} \nu_\delta^{(\lambda, \lambda, \lambda)}.$$

But,

$$\begin{aligned} E &:= 2\tilde{p}'_c \left( \frac{|\tilde{\kappa}|}{2} - \frac{\tilde{\kappa}_1(1+m)+1}{\tilde{p}'_c} + (\tilde{\kappa}_2 - m\tilde{\kappa}_1) \frac{2\tilde{h}^r - 5}{2\tilde{p}'_c} \right) \\ &= |\tilde{\kappa}|(2\tilde{h}^r + 2) - 2(\tilde{\kappa}_1(1+m) + 1) + (\tilde{\kappa}_2 - m\tilde{\kappa}_1)(2\tilde{h}^r - 5) \\ &= \tilde{\kappa}_2(4\tilde{h}^r - 3) + \tilde{\kappa}_1(3m - 2\tilde{h}^r(m-1)) - 2, \end{aligned}$$

where

$$\begin{aligned}\tilde{\kappa}_2(4\tilde{h}^r - 3) &= \frac{4m}{m+1} - 3\tilde{\kappa}_2, \\ \tilde{\kappa}_1(3m - 2\tilde{h}^r(m-1)) &= m\frac{\tilde{\kappa}_1}{\tilde{\kappa}_2}\left(\frac{3}{H} - 2\frac{m-1}{m+1}\right).\end{aligned}$$

Since  $H \geq B = 3$ , we see that  $3/H - 2(m-1)/(m+1) \leq (3-m)/(m+1) \leq 0$  if  $m \geq 3$ . Thus, if  $m \geq 3$ , then since  $\tilde{\kappa}_1/\tilde{\kappa}_2 = 1/a < 1/m$  we see that

$$\tilde{\kappa}_1(3m - 2\tilde{h}^r(m-1)) \geq 3\tilde{\kappa}_2 - 2\frac{m-1}{m+1}$$

and altogether we find that  $E \geq 0$  (even with strict inequality, if  $H > 3$ ). We thus have proved

$$(6.10) \quad \left( \int |\hat{f}|^2 d\mu_{1,k} \right)^{\frac{1}{2}} \leq C \|f\|_{L^{p_c}},$$

with a constant  $C$  not depending on  $k$ , provided that  $m \geq 3$ .

**Assume finally that  $B = 3$  and  $m = 2$ .** Recall also that we are still assuming that  $h^r + 1 > B = 3$  and  $\theta_c \geq 2/7$ , so that  $3 < h^r + 1 \leq 7/2$ .

We shall prove that the Newton-polyhedron of  $\tilde{\phi}^a$  respectively  $\phi$  will have a particular structure. Indeed, if  $\phi$  is analytic, then one can show that  $\phi$  is of type  $Z, E, J, Q$  etc., in the sense of Arnol'd's classification of singularities (compare [3]).

We shall, however, content ourselves with a little less information, which will nevertheless be sufficient for our purposes.

Recall from [21] the notion of *augmented Newton polyhedron*  $\mathcal{N}^r(\tilde{\phi}^a)$  of  $\tilde{\phi}^a$ . If  $L$  denotes the principal line of  $\mathcal{N}(\phi)$ , then it is a supporting line to  $\mathcal{N}(\tilde{\phi}^a)$  too, and if  $(A^+, B^+)$  denotes the right endpoint of the line segment  $L \cap \mathcal{N}(\tilde{\phi}^a)$ , then let  $L^+$  be the half-line  $L^+ \subset L$  contained in the principal line of  $\mathcal{N}(\phi)$  with right endpoint  $(A^+, B^+)$ . Then  $\mathcal{N}^r(\tilde{\phi}^a)$  is the convex hull of the union of  $\mathcal{N}(\tilde{\phi}^a)$  with the half-line  $L^+$ . Recall also that  $\mathcal{N}^r(\tilde{\phi}^a)$  and  $\mathcal{N}^r(\phi^a)$  do agree in the closed half-space above the bi-sectrix  $\Delta$ , so that  $h^r + 1$  is the second coordinate of the point at which the line  $\Delta^{(m)}$  intersects the boundary of  $\mathcal{N}^r(\tilde{\phi}^a)$ .

**Proposition 6.1.** *If  $B = 3$ ,  $m = 2$  and  $3 < h^r + 1 \leq 3.5$ , then  $(A^+, B^+) = (1, 3)$ , and  $\mathcal{N}^r(\tilde{\phi}^a)$  has exactly two edges,  $L^+$  and the line segment  $[(1, 3), (0, n)]$ , which is contained in the principal line  $L^a$  of  $\mathcal{N}(\tilde{\phi}^a)$ .*

*In particular,*

$$(6.11) \quad \kappa = \left(\frac{1}{7}, \frac{2}{7}\right), \quad h^r = d = \frac{7}{3}, \quad \text{and} \quad \tilde{\kappa} = \left(\frac{1}{n}, \frac{n-1}{3n}\right),$$

where  $n > 7$ .

*Proof.* Denote by  $(A', B') := (A'_{(0)}, B'_{(0)}) \in L^a$  the left endpoint of the principal face  $\pi(\tilde{\phi}^a)$  of the Newton polyhedron of  $\tilde{\phi}^a$ . Then  $B' \geq B = 3$ . In a first step, we prove that  $B' = 3$ .

Assume, to the contrary, that we had  $B' \geq 4$  (observe that  $B'$  is an integer). Since the line  $L^a$  has slope strictly less  $1/m = 1/2$ , then it easily seen that the line  $L^a$  would intersect the line  $\Delta^{(2)}$  at some point with second coordinate  $z_2$  strictly bigger than 3.5, so that  $h^r + 1 \geq z_2 > 3.5$ , which would contradict our assumption (Figure 2).

Thus,  $B' = B = 3$ . In a second step, we show that  $A' = 1$ . To this end, let us here work with  $\phi^a$  in place of  $\tilde{\phi}^a$ . Note the point  $(A', B')$  is also the left endpoint of the principal face of  $\mathcal{N}(\phi^a)$ , and that the principal faces of the Newton polyhedra of  $\phi^a$  and  $\tilde{\phi}^a$  both lie on the same line  $L^a$ , since the last step in the change to modified adapted coordinates (3.4) preserves the homogeneity  $\tilde{\kappa}$ . This shows that also  $A' \in \mathbb{N}$ . Moreover,  $A' \geq 1$ , for otherwise we had  $A' = 0$  and thus  $h^r + 1 = 3$ .

Assume that  $A' \geq 2$ . We have to distinguish two cases.

a) If the line  $L$ , which has slope  $1/2$ , contains the point  $(A', B')$ , then the assumption  $A' \geq 2$  would imply that  $h^r + 1 > 3.5$  (see Figure 3).

b) If not, then  $\pi(\phi^a)$  will have an edge  $\gamma = [(A'', B''), (A', B')]$  with right endpoint  $(A', B')$ , and  $L$  must touch  $\mathcal{N}(\phi^a)$  in a point contained in an edge strictly left to  $\gamma$ . But then the line  $L''$  containing  $\gamma$  must have slope strictly less than the slope  $1/2$  of  $L$ , and necessarily  $B'' > B' = 3$ , hence  $B'' \geq 4$ . It is then again easily seen that the line  $L''$  would intersect the line  $\Delta^{(2)}$  at some point with second coordinate  $z_2$  strictly bigger than 3.5, so that again  $h^r + 1 \geq z_2 > 3.5$ , which would contradict our assumption (Figure 3).

We have thus found that  $(A', B') = (1, 3)$ . Assume finally that  $\mathcal{N}(\phi)$  had a vertex  $(A'', B'')$  to the left of  $(1, 3)$ . Then necessarily  $A'' = 0$  and  $B'' \geq 4$ , so that the line passing through  $(A'', B'')$  and  $(1, 3)$  had slope at least 1, a contradiction. We have seen that  $\mathcal{N}(\phi)$  is contained in the half-plane where  $t_1 \geq 1$ , and thus the line  $L$  must pass through the point  $(1, 3)$ , and the claim on the structure of  $\mathcal{N}^r(\tilde{\phi}^a)$  is now obvious.

But then clearly  $\Delta^{(2)}$  will intersect the boundary of  $\mathcal{N}^r(\tilde{\phi}^a)$  in a point of  $L^+$ , so that  $h^r = d$ . The remaining statements in (6.11) are now easily verified. Q.E.D.

With this structural result, we can now conclude the discussion of this case. Indeed, by Proposition 6.1 we have  $\theta_c = 3/10 > 2/7$ , and, arguing as before, only with  $\theta_c$  in place of  $\tilde{\theta}_c$ , we obtain

$$(6.12) \quad \sum_{\lambda \gg 1} \|T_\delta^\lambda\|_{p_c \rightarrow p'_c} \lesssim \delta_0^{\frac{2-7\theta_c}{2}}.$$

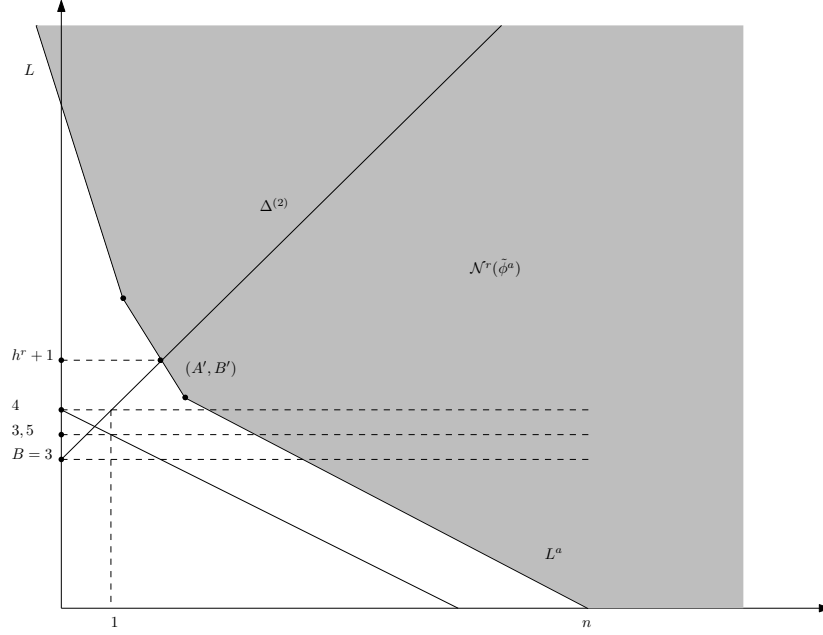


FIGURE 2

Following the previous discussion, we find that the exponent  $E$  in the estimate (6.9) is here given by

$$\begin{aligned} E &:= 2p'_c \left( \frac{|\tilde{\kappa}|}{2} - \frac{\tilde{\kappa}_1(1+2) + 1}{p'_c} + (\tilde{\kappa}_2 - 2\tilde{\kappa}_1) \frac{2h^r - 5}{2p'_c} \right) \\ &= \tilde{\kappa}_2(4h^r - 3) + \tilde{\kappa}_1(6 - 2h^r) - 2. \end{aligned}$$

By means of (6.11) one then computes that

$$E = \frac{19}{9} \frac{n-1}{n} + \frac{4}{3} \frac{1}{n} - 2 = \frac{n-7}{9n} > 0,$$

so that the uniform estimate (6.10) remains valid also in this case.

**6.2. The case where  $h^r + 1 \leq B$ .** In this case, since  $d < h \leq h^r + 1$ , we have  $d < B$ , and since we are assuming that  $d > 5/2$ , we see that we may assume that  $B \geq 3$ .

Moreover, it is obvious from the structure of the Newton polyhedron of  $\phi^a$  that necessarily  $m + 1 \leq B$ , and

$$(6.13) \quad h^r + 1 = h_{i_{\text{pr}}} + 1 = \frac{1 + (m+1)\tilde{\kappa}_1}{|\tilde{\kappa}|}.$$

Indeed, this follows from the geometric interpretation of the notion of  $r$ -height given directly after Remarks 1.3 in [21], since the line  $\Delta^{(m)}$  intersects the principal face  $\pi(\phi^a)$  (see Figure 3).

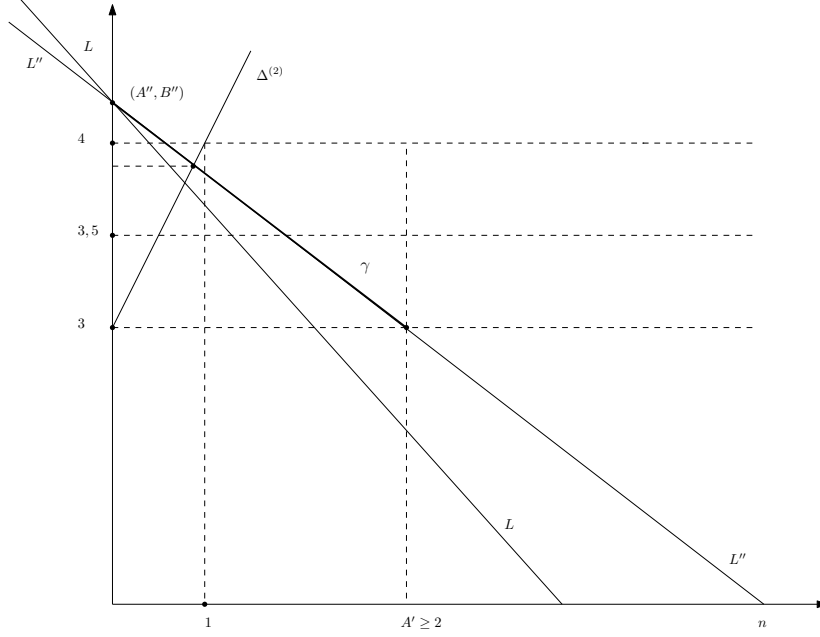


FIGURE 3

Therefore, in passing from the measure  $\nu_\delta$  to  $\mu_k$ , no further gain is possible in this situation in (3.14).

**Consider first Case A.** Corollary 3.2 (b) implies that  $\theta_c \leq \tilde{\theta}_B$ . Thus, by (6.4) we have

$$\|T_\delta^\lambda\|_{p_c \rightarrow p'_c} \lesssim \lambda^{-\frac{1}{3} - \frac{1}{B} + \frac{7}{3} \frac{m+1}{mB+m+1}} = \lambda^{-\frac{M(m,B)}{3B(mB+m+1)}},$$

where

$$M(m, B) := mB^2 - (3m + 6)B + 3(m + 1)$$

is increasing in  $B$  if  $B \geq 3$  and  $m \geq 2$ . Since  $B \geq m + 1$ , we thus have

$$M(m, B) \geq m^3 - m^2 - 5m - 3,$$

and the right-hand side of this inequality is increasing in  $m$  if  $m \geq 2$  and assumes the value 0 if  $m = 3$ . Therefore,  $M(m, B) \geq 0$  if  $m \geq 3$ , even with strict inequality if  $B > m + 1$ .

We thus see that the estimates of  $\|T_\delta^\lambda\|_{p_c \rightarrow p'_c}$  sum for  $m \geq 3$  in  $\lambda \leq \delta_0^{-B/(B-1)}$  when  $B > m + 1$ , and are at least uniform, if  $B = m + 1$ .

Finally, when  $m = 2$ , then  $M(2, B) = 2B^2 - 12B + 9 = 2[(B - 3)^2 - 9/2] > 0$ , iff  $B \geq 6$ . We thus find that

$$(6.14) \quad \sum_{\lambda \leq \delta_0^{-B/(B-1)}} \|T_\delta^\lambda\|_{p_c \rightarrow p'_c} \lesssim 1, \quad \text{except possibly when } m = 2, B = 3, 4, 5, \text{ or } m = 3, B = 4.$$

Moreover, we have

$$\|T_\delta^\lambda\|_{p_c \rightarrow p'_c} \lesssim 1 \quad \text{if } m = 3, B = 4.$$

But, recall that  $\theta_c < \tilde{\theta}_c \leq \tilde{\theta}_B$ , unless  $4 = B = H = h^r + 1 = d + 1$  here, so that  $\theta_c = 1/4$ , so that in the case where we can only obtain the previous uniform estimate for the  $T_\delta^\lambda$ , we will have  $\theta_c = 1/4$ . Moreover, by (6.2) and (6.3) we have  $\|\nu_\delta^\lambda\|_\infty \lesssim \lambda^{7/4}$ , since we are in Case A, and  $\|\widehat{\nu_\delta^\lambda}\|_\infty \lesssim \lambda^{-7/12}$ , where  $-(1 - \theta_c)7/12 + \theta_c 7/4 = 0$ . We may thus again estimate the operator of convolution with the Fourier transform of the complex measure  $\sum_{\lambda \leq \delta_0^{-4/3}} \nu_\delta^\lambda$  by means of the real-interpolation Proposition 5.1, in the same way as we did in the corresponding case where  $m = 2$  and  $B = 3$ .

We are thus left with the cases where  $m = 2, h^r + 1 \leq B$  and  $B = 3, 4, 5$ . So assume in the sequel that that  $m = 2$  and  $h^r + 1 \leq B$

If  $m = 2$  and  $B = 3$ , then  $B = m + 1$ , and since we are assuming  $h^r + 1 \leq B$ , a look at the Newton polyhedron shows that necessarily  $H = B = 3 = h^r + 1$ .

**Lemma 6.2.** *Assume that  $m = 2$  and  $B = 4, 5$ . Then we have*

$$-\frac{1}{3} - \frac{1}{B} + \frac{7}{3}\tilde{\theta}_c < 0,$$

provided

$$(6.15) \quad H > H(B) := \begin{cases} \frac{9}{2}, & \text{if } B = 4, \\ \frac{81}{16}, & \text{if } B = 5. \end{cases}$$

*Proof.* For  $m = 2$  we have

$$-\frac{1}{3} - \frac{1}{B} + \frac{7}{3}\tilde{\theta}_c = -\frac{1}{3} - \frac{1}{B} + \frac{7}{2H+3} < 0$$

if and only if

$$(6.16) \quad H > \frac{21}{2} \frac{B}{B+3} - \frac{3}{2},$$

and it is easily checked that this holds true if and only if when  $H \geq H(B)$  when  $B = 4, 5$ . Q.E.D.

Since by Corollary 3.2 (b)  $\theta_c \leq \tilde{\theta}_c$ , the previous lemma shows that for  $m = 2$ , (6.14) can be sharpened as follows:

$$(6.17) \quad \sum_{\lambda \leq \delta_0^{-B/(B-1)}} \|T_\delta^\lambda\|_{p_c \rightarrow p'_c} \lesssim 1,$$

with the possible exceptions when  $B = H = 3 = h^r + 1$ , or  $B = 4, 5, h^r + 1 \leq B$  and  $H \leq H(B)$ .

**Consider finally Case B.** Then, since  $\theta_c \leq \tilde{\theta}_B$ ,

$$\|T_\delta^\lambda\|_{p_c \rightarrow p'_c} \lesssim \lambda^{-\frac{1}{3} - \frac{1}{B} + (\frac{4}{3} + \frac{1}{B}) \frac{m+1}{mB+m+1}} \delta_0^{-\tilde{\theta}_B} = \lambda^{-\frac{B(mB-3)}{3B(mB+m+1)}} \delta_0^{-\tilde{\theta}_B}.$$

The exponent of  $\lambda$  is negative, so we can sum these estimates in  $\lambda$  and obtain

$$\sum_{\lambda > \delta_0^{-B/(B-1)}} \|T_\delta^\lambda\|_{p_c \rightarrow p'_c} \lesssim \delta_0^{[\frac{1}{3} + \frac{1}{B} - (\frac{4}{3} + \frac{1}{B}) \tilde{\theta}_B] \frac{B}{B-1} - \tilde{\theta}_B} = \delta_0^{\frac{B+3-7B\tilde{\theta}_B}{3(B-1)}} = \delta_0^{\frac{M(m,B)}{3(B-1)(mB+m+1)}}.$$

But, our previous discussion of  $M(m, B)$  shows that  $M(m, B) \geq 0$ , unless  $m = 2$  and  $B = 3, 4, 5$  (notice that the case  $m = 3, B = 4$  still works here ).

In the latter cases, we can improve our estimates again by using  $\tilde{\theta}_c$  in place of  $\tilde{\theta}_B$ . Indeed, notice that the condition  $B + 3 - 7B\tilde{\theta}_c > 0$  is equivalent to (6.16), which by Lemma 6.2 does hold true if  $H \leq H(B)$ .

We thus find that

$$(6.18) \quad \sum_{\lambda > \delta_0^{-B/(B-1)}} \|T_\delta^\lambda\|_{p_c \rightarrow p'_c} \lesssim 1,$$

unless  $m = 2$  and  $B = H = 3 = h^r + 1$ , or  $B = 4, 5, h^r + 1 \leq B$  and  $H \leq H(B)$ .

Finally, we observe that the following consequence of Lemma 6.2:

**Corollary 6.3.** *Assume that  $m = 2$ , and that  $B = H = 3$ , or  $B = 4, 5, h^r + 1 \leq B$  and  $H \leq H(B)$ . Then the left endpoint of the principal face of the Newton polyhedron of  $\tilde{\phi}^a$  is of the form  $(A, B)$ , where  $A \in \{0, 1, \dots, B-3\}$ , and we have*

$$\tilde{\kappa} = \left( \frac{1}{n}, \frac{n-A}{Bn} \right) \quad \text{and} \quad h^r + 1 = \frac{n+3}{n+B-A} B,$$

where  $n$  must satisfy  $n > 2B$  and  $(Bn)/(n-A) = H \leq H(B)$ . Moreover,  $\tilde{\phi}^a$  is smooth here, so that in particular  $a$  and  $n$  are integers.

*Proof.* Adopting the notation for the proof of Proposition 6.1, let  $(A', B')$  denote the left endpoint of  $\pi(\phi^a)$ , which, as we recall, is also the left endpoint of  $\pi(\phi^a)$ . If  $B = 4, 5$ , then  $B \leq B' \leq H \leq H(B) < B+1$ , so that  $B' = B$ . For  $B = 3 = H$ , the same conclusion applies. Next, since we assume that  $h^r + 1 \leq B$ , and since the line  $\Delta^{(2)}$  intersects the line where  $t_2 = B$  in the point  $(B-3, B)$ , we must have  $0 \leq A' \leq B-3$ . This implies the first statement of the corollary, because  $A'$  is integer.



The remaining statements, except for the last one, follow easily (for the identity for  $h^r + 1$ , recall (6.13)).

Finally, in order to see that in these cases we must have  $a, n \in \mathbb{N}$ , observe that  $\tilde{\phi}_\kappa^a$  must be of the form

$$\tilde{\phi}_\kappa^a(x) = x_1^A x_2^B + c_1 x_1^n, \quad c_1 \neq 0.$$

But then  $\phi_\kappa^a$  will be given by

$$\phi_\kappa^a(x) = x_1^A (x_2 + c_0 x_1^a)^B + c_1 x_1^n$$

(compare (3.3)). This must be a polynomial in  $(x_1, x_2)$ , since  $\phi^a$  is smooth. Expanding  $(x_2 + c_0 x_1^a)^B$ , it is clear that  $a$  must be an integer. But then, necessarily also  $\tilde{\phi}^a$  is smooth, which implies that  $n$  must be an integer too.

Q.E.D.

**6.3. The case where  $B = 5$ .** The case where  $B = 5$  can now be treated quite easily. Indeed, going back to the estimations in Subsection 6.2, recall from (6.4) that we have the estimate

$$\|T_\delta^\lambda\|_{p_c \rightarrow p'_c} \lesssim \lambda^{-\frac{1}{3} - \frac{1}{5} + \frac{7\theta_c}{3}}.$$

This estimate is valid in Case A as well as in Case B (in the latter case, an even stronger estimate is valid, but we won't need it). And, according to Corollary 6.3, if  $B = 5$  then we have the precise formula

$$\theta_c = \frac{n + 5 - A}{5(n + 3)}$$

for  $\theta_c$ , where  $A \in \{0, 1, 2\}$  and  $n > 2B = 10$ . Since  $n$  is here an integer, we find that  $n \geq 11$ , and thus  $\theta_c \leq (11 + 5)/70 = 8/35$ , with strict inequality, unless  $A = 0$  and  $n = 11$ . This shows that the exponent of  $\lambda$  in the estimate for  $\|T_\delta^\lambda\|_{p_c \rightarrow p'_c}$  is strictly negative, so that we can sum these estimates over all dyadic  $\lambda \geq 1$ , unless  $A = 0$  and  $n = 11$ , where we get a uniform estimate

$$(6.19) \quad \|T_\delta^\lambda\|_{p_c \rightarrow p'_c} \lesssim 1.$$

However, if  $A = 0, B = 5$  and  $n = 11$ , then  $\theta_c = 8/35$ , and moreover, by (6.2) and (6.3), we have  $\|\nu_\delta^\lambda\|_\infty \lesssim \lambda^{9/5}$  and  $\|\widehat{\nu_\delta^\lambda}\|_\infty \lesssim \lambda^{-8/15}$ , where  $-(1 - \theta_c)8/15 + \theta_c 9/5 = 0$ . We may thus again estimate the operator of convolution with the Fourier transform of the complex measure  $\sum_{\lambda \gg 1} \nu_\delta^\lambda$  by means of the real-interpolation Proposition 5.1, by adding this measure to the list of measures  $\mu^i, i \in I$ , of the second class in (5.8).

**6.4. The remaining cases.** The case where  $m = 2$  and  $B = 3$  has essentially already been addressed in the previous subsection. Its treatment will require a refined Airy type analysis, following ideas from Section 6 of [21], but the discussion will be even more involved than in [21], where we have had  $m = 2$  and  $B = 2$ . Also in the case  $m = 2$  and  $B = 4$ , we shall need some Airy type analysis, but of simpler form.

Combining all the previous estimates and applying (3.14), we see that we have proved

**Corollary 6.4.** *Assume that  $m$  and  $H, B$  are not such that  $m = 2$  and  $B = H = 3 = h^r + 1$ , or  $m = 2, B = 4, h^r + 1 \leq 4$  and  $H \leq H(4)$ . Then the estimates in Proposition 2.2 hold true for  $l = 1$ .*

More precisely, in view of Corollary 6.3, open are the following situations:

$$(6.20) \quad m = 2, B = 3, 4, h^r + 1 \leq B, \quad \text{and } \lambda_1 \sim \lambda_2 \sim \lambda_3,$$

and the left endpoint of the principal face of the Newton polyhedron of  $\tilde{\phi}^a$  is of the form  $(A, B)$ , where  $A \in \{0, \dots, B - 3\}$ .

In these situations, we have

$$(6.21) \quad \tilde{\kappa} = \left( \frac{1}{n}, \frac{n - A}{Bn} \right) \quad \text{and} \quad h^r + 1 = \frac{n + 3}{n + B - A} B,$$

where  $n$  must be an integer satisfying  $n > 2B$ , and  $(Bn)/(n - A) = H \leq H(B)$ , if  $B = 4$ .

The discussion of these cases will occupy the major part of remainder of this article. Before we come to this, let us first study also the contributions by the remaining domains  $D'_{(l)}$ ,  $l \geq 2$ , which will turn out to be simpler.

## 7. RESTRICTION ESTIMATES FOR THE DOMAINS $D'_{(l)}$ , $l \geq 2$

For the domains  $D'_{(l)}$ ,  $l \geq 2$ , we can essentially argue as in the preceding section, by putting here

$$(7.1) \quad \tilde{\phi}^a := \phi^{(l+1)}, \quad \tilde{\kappa} := \kappa^{(l)}, \quad D^a := D_{(l)}^a, \quad L^a := L_{(l)}, \quad \text{etc..}$$

$H$  and  $n$  are defined correspondingly.

We then have the following analogue of Lemma 3.1.

**Lemma 7.1.** (a) *If  $l \geq 2$ , then  $p'_c > \tilde{p}'_c$ .*  
 (b) *If  $m \geq 3$  and  $H \geq 2$ , or  $m = 2$  and  $H \geq 3$ , then*

$$\tilde{p}'_c \geq p'_H \geq p'_B.$$

*Proof.* (a) We can follow the proof of Lemma 3.1 (a) and see again that  $p'_c > \tilde{p}'_c$ , unless the principal face  $\pi(\tilde{\phi}^a)$  of  $\tilde{\phi}^a$  is the edge  $[(0, H), (n, 0)]$ . However, for  $l \geq 2$ , we know from Section 2 that  $\pi(\tilde{\phi}^a)$  is the edge  $\gamma'_{(l)}$ , which lies below the bi-sectrix  $\Delta$ , so that we cannot have  $p'_c = \tilde{p}'_c$ .

(b) follows as before. Q.E.D.

This implies the following, stronger analogue of Corollary 3.2.

**Corollary 7.2.** *Assume that  $l \geq 2$ .*

- (a) *If  $m \geq 3$  and  $H \geq 2$ , or  $m = 2$  and  $H \geq 3$ , then  $\theta_c < \theta_B$ .*
- (b) *If  $h^r + 1 \leq B$ , then  $\theta_c < \tilde{\theta}_c$ , unless  $B = H = h^r + 1 = d + 1$ , where  $\theta_c = \tilde{\theta}_c$ .*
- (c) *If  $H \geq 3$ , then  $\theta_c < 1/3$ , unless  $H = 3$  and  $m = 2$ .*

Finally, in place of Proposition 6.1, we have

**Proposition 7.3.** *Assume that  $l \geq 2$ , and that  $B = 3$ ,  $m \geq 2$ . Then  $h^r + 1 > 3.5$ .*

*Proof.* Assume we had  $h^r + 1 \leq 3.5$ , and denote by  $(A', B') := (A'_{(l-1)}, B'_{(l-1)}) \in L^a$  the left endpoint of the principal face  $\pi(\tilde{\phi}^a) = \gamma'_{(l)}$  of the Newton polyhedron of  $\tilde{\phi}^a$ . Then  $B' \geq B = 3$ . Arguing in the same way as in the first step of the proof of Proposition 3.2, we see that  $B' = B = 3$ .

Then the preceding edge  $\gamma'_{(l-1)}$  will have a left endpoint  $(A'', B'')$  with  $B'' \geq 4$ . Moreover, since the line  $L^{(1)}$  has slope  $1/a < 1/m \leq 1/2$ , the line  $L^{(l-1)}$  containing  $\gamma'_{(l-1)}$  has slope strictly less than  $1/2$ . But then it will intersect the line  $\Delta^{(2)}$  at some point with second coordinate  $z_2$  strictly bigger than 3.5, so that we would again arrive at  $h^r + 1 \geq z_2 > 3.5$ , which contradicts our assumption. Q.E.D.

These results allow us to proceed exactly as in Sections 4, 6, even with some simplifications. Indeed, a careful inspection of our arguments in these sections reveals that here all the series which appear do sum, and no further interpolation arguments are required.

This is because of the stronger estimate  $p'_c > \tilde{p}'_c$  of Lemma 7.1 and the stronger statement of Proposition 7.3, which implies in particular that  $\theta_c < 1/3$  when  $B = 3$ .

Moreover, in Subsection 6.1, the more delicate case where  $B = 3$  and  $\theta_c \geq 2/7$  does not appear anymore, because by Proposition 7.2 we have  $\theta_c < 2/7$ .

Observe finally that if  $m = 2$ ,  $B = 3, 4, 5$ ,  $h^r + 1 \leq B$  and  $H \leq H(B)$ , then Corollary 6.3, whose proof applies equally well when  $l \geq 2$ , shows that left endpoint of the principal face of the Newton polyhedron of  $\tilde{\phi}^a$  is of the form  $(A, B)$ , where  $A \leq B - 3$ . However, if  $l \geq 2$ , this endpoint must lie on or below the bi-sectrix, which leads to a contraction. These cases therefore cannot arise when  $l \geq 2$ .

We therefore obtain

**Corollary 7.4.** *The estimates in Proposition 2.2 hold true for every  $l \geq 2$ .*

## 8. THE REMAINING CASES WHERE $m = 2$ AND $B = 3$ OR $B = 4$ : PRELIMINARIES

We finally turn to the discussion of the remaining cases which are described by (6.20) and (6.21). In view of our definition of  $B$  (cf. (3.5)), we see that in these cases  $Q(y_1, y_2) = c_0 y_1^A$ , where  $c_0 \neq 0$  and  $A \in \{0, \dots, B - 3\}$ . Indeed, this is obvious if  $A = 0$ , hence in particular if  $B = 3$ , and if  $B = 4$  and  $A = 1$ , then our assumption that  $H \leq H(4) < 5$  implies that the Taylor support of  $\tilde{\phi}_\kappa^a$  cannot contain a point on the second coordinate axis. Let us assume without loss of generality that  $c_0 = 1$ , so that by (3.5)

$$(8.1) \quad \tilde{\phi}_\kappa^a(x) = x_1^A x_2^B + c_1 x_1^n, \quad c_1 \neq 0.$$

Then, by (3.9), (3.10) and (3.12), we may write

$$(8.2) \quad \phi_\delta(x) := x_2^B b(x_1, x_2, \delta) + x_1^n \alpha(\delta_1 x_1) + r(x_1, x_2, \delta), \quad (x_1 \sim 1, |x_2| < \varepsilon),$$

where  $b(x_1, x_2, 0) = x_1^A \sim 1$ ,  $\alpha(0) \neq 0$  and

$$r(x_1, x_2, \delta) = \sum_{j=1}^{B-1} x_2^j x_1^{n_j} \delta_{j+2} \alpha_j(\delta_1 x_1).$$

Moreover, either  $\alpha_j(0) \neq 0$ , and then  $n_j$  is fixed (the type of the finite type function  $b_j$ ), or  $\alpha_j(0) = 0$ , and then we may assume that  $n_j$  is as large as we please.

Observe also that if  $A = 0$  (this is necessarily so if  $B = 3$ ), then we have  $Q(x) \equiv 1$  in (3.5), so that  $\tilde{\kappa}_2 = 1/B$ , and consequently

$$(8.3) \quad b(x_1, x_2, \delta) = b_B(\delta_1 x_1, \delta_2 x_2)$$

in (3.9).

Recall also from Corollary 6.3 that here  $\tilde{\phi}^a$  is even smooth, not only fractionally smooth, and that  $a$  and  $n$  are integers.

**8.1. The fine structure of the phase  $\phi_\delta$ .** We shall need to derive more specific information on the phase  $\phi_\delta$ , and begin with the “error term”  $r(x_1, x_2, \delta)$ .

**Corollary 8.1.** *By some slight change of coordinates, we may even assume that the term with index  $j = B - 1$  vanishes, i.e., that*

$$(8.4) \quad r(x_1, x_2, \delta) = \sum_{j=1}^{B-2} x_2^j x_1^{n_j} \delta_{j+2} \alpha_j(\delta_1 x_1).$$

*In particular, in view of (3.11), we may assume that  $\delta = (\delta_0, \delta_1, \delta_2, \delta_3, \dots, \delta_B)$  is given by*

$$(8.5) \quad \delta := (2^{-k(\tilde{\kappa}_2 - 2\tilde{\kappa}_1)}, 2^{-k\tilde{\kappa}_1}, 2^{-k\tilde{\kappa}_2}, 2^{-k(n_1\tilde{\kappa}_1 + \tilde{\kappa}_2 - 1)}, \dots, 2^{-k(n_{B-2}\tilde{\kappa}_1 + (B-2)\tilde{\kappa}_2 - 1)}).$$

*Moreover, the complete phase corresponding to  $\phi_\delta$  is given by*

$$\Phi(x, \delta, \xi) := \xi_1 x_1 + \xi_2(\delta_0 x_2 + x_1^m \omega(\delta_1 x_1)) + \xi_3 \phi_\delta(x_1, x_2).$$

*Proof.* Indeed, going back to the domain  $D^a = \{(y_1, y_2) : 0 < y_1 < \varepsilon, |y_2| < \varepsilon y_1^a\}$ , observe that (8.1) implies that  $\partial_1^A \partial_2^{B-1} \tilde{\phi}_\kappa^a(y_1, 0) \equiv 0$ , whereas  $\partial_1^A \partial_2^B \tilde{\phi}_\kappa^a(y_1, 0) \equiv A!B! \neq 0$  for  $|y_1| < \varepsilon$ . Moreover, since  $(A, B)$  is a vertex of  $\mathcal{N}(\tilde{\phi}^a)$ , we also have  $\partial_1^A \partial_2^B \tilde{\phi}^a(0, 0) = A!B! \neq 0$ , whereas  $\partial_1^A \partial_2^{B-1} \tilde{\phi}^a(0, 0) = 0$ .

Then the implicit function theorem implies that, that for  $\varepsilon$  sufficiently small, there is a smooth function  $\rho(y_1)$ ,  $-\varepsilon < y_1 < \varepsilon$ , such

$$\partial_1^A \partial_2^{B-1} \tilde{\phi}^a(y_1, \rho(y_1)) \equiv 0.$$

and comparing  $\tilde{\kappa}$ -principal parts, it is easy to see that the  $\tilde{\kappa}$ -principal part of  $\rho$  has  $\tilde{\kappa}$ -degree strictly bigger than the degree of  $y_1^a$ . Thus, if we perform the further change of coordinates  $(z_1, z_2) := (y_1, y_2 - \rho(y_1))$ , in which  $\phi$  is represented, say, by  $\tilde{\phi}$ , it is easily seen that the Newton polyhedra of  $\tilde{\phi}^a$  and  $\tilde{\phi}$  as well as their  $\tilde{\kappa}$ -principal parts are the same (cf. similar arguments in [18]). Replacing  $\psi^{(2)}(y_1)$  by  $\psi^{(2)}(y_1) + \rho(y_1)$  and

modifying  $\omega(y_1)$  accordingly, we then find that in the corresponding modified adapted coordinates  $z_1 = x_1, z_2 = x_2 - (\psi^{(2)}(x_1) + \rho(x_1))$ , the function  $\tilde{\phi}$  satisfies

$$(8.6) \quad \partial_1^A \partial_2^{B-1} \tilde{\phi}(z_1, 0) \equiv 0.$$

On the other hand, like  $\tilde{\phi}^a$ , it must be of the form

$$\tilde{\phi}(z_1, z_2) = z_2^B b_B(z_1, z_2) + z_1^n \alpha(z_1) + \sum_{j=1}^{B-1} z_2^j b_j(z_1),$$

where  $b_B(z_1, z_2) = z_1^A$ , except for terms of  $\tilde{\kappa}$ -degree strictly bigger than  $\tilde{\kappa}_1 A$ , and thus (8.6) implies that  $b_{B-1}^{(A)}(z_1) \equiv 0$ , i.e.,  $b_{B-1}(z_1) = \sum_{k=0}^{A-1} a_k z_1^k$ .

The corresponding terms  $z_2^{B-1} a_k z_1^k$  in  $z_2^{B-1} b_{B-1}(z_1)$  have  $\tilde{\kappa}$ -degree  $k\tilde{\kappa}_1 + (B-1)\tilde{\kappa}_2 \leq (A-1)\tilde{\kappa}_1 + (B-1)\tilde{\kappa}_2 < A\tilde{\kappa}_1 + B\tilde{\kappa}_2 = 1$ , and thus must all vanish, since they should also have  $\tilde{\kappa}$ -degree strictly bigger than 1, as required in (3.6). We thus find that

$$\tilde{\phi}(z_1, z_2) = z_2^B b_B(z_1, z_2) + z_1^n \alpha(z_1) + \sum_{j=1}^{B-2} z_2^j b_j(z_1),$$

which, after re-scaling, implies (8.4).

Q.E.D.

Recall also that we are interested in the frequency domains where  $|\xi_j| \sim \lambda_j$ ,  $j = 1, 2, 3$ , assuming that  $\lambda_1 \sim \lambda_2 \sim \lambda_3$ .

We shall assume for the sake of simplicity that

$$\lambda_1 = \lambda_2 = \lambda_3 \gg 1.$$

With some slight change of notation, compared to Section 4, we shall here put  $\lambda := \lambda_1 = \lambda_2 = \lambda_3$ , and write accordingly

$$\widehat{\nu_\delta^\lambda}(\xi) := \chi_1\left(\frac{\xi_1}{\lambda}\right) \chi_1\left(\frac{\xi_2}{\lambda}\right) \chi_1\left(\frac{\xi_3}{\lambda}\right) \int e^{-i\Phi(y, \delta, \xi)} \eta(y) dy,$$

i.e.,

$$\nu_\delta^\lambda(x) = \lambda^3 \int \check{\chi}_1\left(\lambda(x_1 - y_1)\right) \check{\chi}_1\left(\lambda(x_2 - \delta_0 y_2 - y_1^2 \omega(\delta_1 y_1))\right) \check{\chi}_1\left(\lambda_3(x_3 - \phi_\delta(y))\right) \eta(y) dy.$$

Recall also that  $\text{supp } \eta \subset \{x_1 \sim 1, |x_2| < \varepsilon\}$ , ( $\varepsilon \ll 1$ ).

As before, we change coordinates from  $\xi = (\xi_1, \xi_2, \xi_3)$  to  $s_1, s_2$  and  $s_3 := \xi_3/\lambda$ , by writing  $\xi = \xi(s, \lambda)$ , with

$$\xi_1 = s_1 \xi_3 = \lambda s_1 s_3, \quad \xi_2 = s_2 \xi_3 = \lambda s_2 s_3, \quad \xi_3 = \lambda s_3.$$

Accordingly, we write

$$(8.7) \quad \Phi(y, \delta, \xi) = \lambda s_3 \left( \Phi_1(y_1, \delta_1, s) + s_2 \delta_0 y_2 + y_2^B b(y_1, y_2, \delta) + r(y_1, y_2, \delta) \right),$$

where

$$\Phi_1(y_1, \delta_1, s) := s_1 y_1 + s_2 y_1^2 \omega(\delta_1 y_1) + y_1^n \alpha(\delta_1 y_1).$$

Notice that here  $|s_j| \sim 1$ ,  $j = 1, 2, 3$ , and we have put  $s := (s_1, s_2)$ . By passing from  $s_j$  to  $-s_j$ , if necessary, we may and shall in the sequel always assume that

$$s_j \sim 1, \quad j = 1, 2, 3$$

(notice that these changes of signs may cause a change of sign of the  $x_j$  and  $\omega$ , respectively of  $\Phi$ ).

Let us fix  $s^0$  in this domain, and consider the phase function  $\Phi_1(x_1, 0, s^0)$  for  $\delta_1 = 0$ . In case that this phase has at worst non-degenerate critical points  $x_1^c \sim 1$ , then the same is true for sufficiently small  $\delta_1$ , and the estimate for  $\widehat{\nu_\delta^\lambda}$  in Section 6 can be improved to

$$\|\widehat{\nu_\delta^\lambda}\|_\infty \lesssim \lambda^{-\frac{1}{2}-\frac{1}{B}},$$

and thus in Case A we obtain the better estimate

$$\|T_\delta^\lambda\|_{p_c \rightarrow p'_c} \lesssim \lambda^{-\frac{1}{2}-\frac{1}{B}+\frac{5}{2}\theta_c},$$

compared to (6.4) (of course,  $T_\delta^\lambda$  stands here for  $T_\delta^{(\lambda, \lambda, \lambda)}$ ). Moreover, by means of (6.20), (6.21), one checks easily that the exponent of  $\lambda$  in this estimate is strictly negative, if  $B = 4$ , and zero, if  $B = 3$ . Thus we can sum these estimate over all dyadic  $\lambda \gg 1$  if  $B = 4$ , and obtain at least a uniform estimate when  $B = 3$ . It is easy to see that this case can then still be treated by means of the real interpolation Proposition 5.1, since the relevant frequencies will here be restricted essentially to cuboids in the  $\xi$ -space.

In Case B, where  $\lambda > \delta_0^{-B/(B-1)}$ , we obtain the better estimate

$$\|T_\delta^\lambda\|_{p_c \rightarrow p'_c} \lesssim \lambda^{-\frac{1}{2}-\frac{1}{B}+(\frac{3}{2}+\frac{1}{B})\theta_c} \delta_0^{-\theta_c},$$

compared to (6.5). The exponent of  $\lambda$  is strictly negative (compare the discussion leading to (6.18)), so summing over all  $\lambda > \delta_0^{-B/(B-1)}$ , we find that

$$\sum_{\lambda > \delta_0^{-B/(B-1)}} \|T_\delta^\lambda\|_{p_c \rightarrow p'_c} \lesssim \delta_0^{\frac{B+2-5B\theta_c}{2B-2}}.$$

And, since  $\theta_c \leq \tilde{\theta}_B$ , one easily checks that the exponent of  $\delta_0$  in this estimate is non-negative if  $B \geq 3$ .

We are thus left with the more subtle situation where the phase function  $\Phi_1(x_1, 0, s^0)$  has a degenerate critical point  $x_1^c$  of Airy type. Denoting by  $\Phi'_1, \Phi''_1$  etc. derivatives with respect to  $x_1$ , and arguing as in Section 6 of [21], we see by the implicit function theorem that for  $s$  sufficiently close to  $s^0$  and  $\delta$  sufficiently small, there is a unique, non-degenerate critical point  $x_1^c = x_1^c(\delta_1, s_2) \sim 1$  of  $\Phi'_1$ , i.e.,

$$\Phi''_1(x_1^c(\delta_1, s_2), \delta_1, s) = 0, \quad |s - s^0| < \varepsilon, |\delta| < \varepsilon,$$

if  $\varepsilon$  is sufficiently small. We then shift this critical point to the origin, by putting

$$(8.8) \quad \Phi^\#(x, \delta, \xi) := \frac{1}{s_3 \lambda} \Phi(x_1^c(\delta_1, s_2) + x_1, x_2, \delta, \xi), \quad |(x_1, x_2)| \ll 1$$

(notice that we may indeed assume that  $|(x_1, x_2)| \ll 1$ , since away from  $x_1^c$ , we have at worst non-degenerate critical points, and the previous argument applies).

From Lemma 6.2 in [21] (with  $\beta := \alpha, \sigma = 1, \delta_3 = 0$  and  $b_0 = 0$ ), we then immediately get, after scaling in  $x_1$  so that we may assume that

$$(8.9) \quad -\frac{2\omega(0)}{n(n-1)\alpha(0)} = 1,$$

the following result:

**Lemma 8.2.**  $\Phi^\sharp$  is of the form

$$(8.10) \quad \Phi^\sharp(x, \delta, \xi) = x_1^3 B_3(s_2, \delta_1, x_1) - x_1 B_1(s, \delta_1) + B_0(s, \delta_1) + \phi^\sharp(x, \delta, s_2),$$

with

$$(8.11) \quad \begin{aligned} \phi^\sharp(x, \delta, s_2) &:= x_2^B b(x, \delta, s_2) + \sum_{j=2}^{B-2} \delta_{j+2} x_2^j \tilde{\alpha}_j(x_1, \delta, s_2) \\ &+ x_2 \left( s_2 \delta_0 + \delta_3 (x_1^c(\delta_1, s_2) + x_1)^{n_1} \alpha_1(\delta_1 (x_1^c(\delta_1, s_2) + x_1)) \right), \end{aligned}$$

and where the following hold true:

The functions  $b$  and  $\tilde{\alpha}_j$  are smooth,  $b(x, \delta, s_2) \sim 1$ , and also  $|\tilde{\alpha}_j| \sim 1$ , unless  $\alpha_j$  is a flat function. Moreover,  $B_0, B_1$  and  $B_3$  are smooth functions, and

$$B_3(s_2, \delta_1, 0) = s_2^{\frac{n-3}{n-2}} G_4(\delta_1 s_2^{\frac{1}{n-2}}),$$

where

$$G_4(0) = \frac{n(n-1)(n-2)}{6} \alpha(0).$$

Furthermore, we may write

$$(8.12) \quad \begin{cases} x_1^c(\delta_1, s_2) &= s_2^{\frac{1}{n-2}} G_1(\delta_1 s_2^{\frac{1}{n-2}}), \\ B_0(s, \delta_1) &= s_1 s_2^{\frac{1}{n-2}} G_1(\delta_1 s_2^{\frac{1}{n-2}}) - s_2^{\frac{n}{n-2}} G_2(\delta_1 s_2^{\frac{1}{n-2}}), \\ B_1(s, \delta_1) &= -s_1 + s_2^{\frac{n-1}{n-2}} G_3(\delta_1 s_2^{\frac{1}{n-2}}), \end{cases}$$

where

$$(8.13) \quad \begin{cases} G_1(0) &= 1, \\ G_2(0) &= \frac{n^2-n-2}{2} \alpha(0), \\ G_3(0) &= n(n-2) \alpha(0). \end{cases}$$

Notice that all the numbers in (8.13) are non-zero, since we assume  $n > 2B > 5$ . Finally, if we also write  $G_5 := G_1 G_3 - G_2$ , then we have

$$(8.14) \quad G_3(0) \neq 0, \quad G_5(0) \neq 0.$$

Observe that we obtain here a more specific dependency of  $x_1^c, B_0, B_1$  and  $B_3$  on  $\delta_1$  and  $s_2$  than in [21], due to the fact that the equation for the critical point depends only on the parameter  $\delta_1^{n-2}s_2$  in the coordinate  $y_1 := \delta_1 x_1$ .

Nevertheless, with a slight abuse of notation, we shall frequently also use the short-hand notation  $G_j(s_2, \delta)$  in place of  $G_j(\delta_1 s_2^{\frac{1}{n-2}})$ ,  $j = 1, \dots, 4$

We also remark that the part of the measure  $\nu_\delta$  corresponding to the region described in (8.8), which we shall simply again denote by  $\nu_\delta$ , is given by an expression for its Fourier transform of the form

$$(8.15) \quad \widehat{\nu}_\delta(\xi) = \int e^{-is_3 \lambda \Phi^\sharp(x, \delta, \xi)} a(x, \delta, s) dx \quad (\text{with } \xi = \xi(s, \lambda)),$$

where  $a$  is a smooth function with compact support in  $x$  such that  $|x| \leq \varepsilon$  on  $\text{supp } a$ .

**Remark 8.3.** *It will be important in the sequel to observe that every single  $\delta_j$  is a fractional power of  $2^{-k}$ , hence also of  $\delta_0$ , i.e., there is some positive rational number  $\tau > 0$  such that  $\delta_j = \delta_0^{q_j \tau}$ , with positive integers  $q_j$  (cf. (8.5)).*

In the sequel, we shall need more precise information on the structure of the last term of  $\phi^\sharp$  in (8.11):

**Lemma 8.4.** *Let*

$$a_1(x_1, \delta, s_2) := s_2 \delta_0 + \delta_3 (x_1^c(\delta_1, s_2) + x_1)^{n_1} \alpha_1(\delta_1 (x_1^c(\delta_1, s_2) + x_1))$$

*be the coefficient of  $x_2$  in the last term of  $\phi^\sharp$  in (8.11). Then  $a_1$  can be re-written in the form*

$$(8.16) \quad a_1(x_1, \delta, s_2) = \delta_{3,0} \tilde{\alpha}_1(x_1, \delta_0^\tau, s_2) + \delta_0 x_1 \alpha_{1,1}(x_1, \delta_0^\tau, s_2),$$

*with smooth functions  $\tilde{\alpha}_1$  and  $\alpha_{1,1}$ , and where  $\delta_{3,0}$  is of the form  $\delta_{3,0} = \delta_0^{q_{3,0} \tau}$ , with some positive integer  $q_{3,0}$ . Moreover, two possible cases may arrive:*

**Case ND: The non-degenerate case:**  $\alpha_{1,1} \equiv 0$ ,  $|\tilde{\alpha}_1| \sim 1$  and  $\delta_{3,0} = \max\{\delta_0, \delta_3\} \geq \delta_0$ ,

*or*

**Case D: The degenerate case:**  $|\alpha_{1,1}| \sim 1$ ,  $\tilde{\alpha}_1 = \tilde{\alpha}_1(\delta_0^\tau, s_2)$  is independent of  $x_1$ ,  $\delta_0 = \delta_3$  and  $\delta_{3,0} \ll \delta_0$ . Moreover, either  $|\tilde{\alpha}_1| \sim 1$ , or we can choose  $q_{3,0} \in \mathbb{N}$  as large as we wish.

*In particular, we may write*

$$(8.17) \quad \begin{aligned} \phi^\sharp(x, \delta, s) &= x_2^B b(x, \delta_0^\tau, s_2) + \sum_{j=2}^{B-2} \delta_{j+2} x_2^j \tilde{\alpha}_j(x_1, \delta_0^\tau, s_2) \\ &+ \delta_{3,0} x_2 \tilde{\alpha}_1(x_1, \delta_0^\tau, s_2) + \delta_0 x_1 x_2 \alpha_{1,1}(x_1, \delta_0^\tau, s_2), \end{aligned}$$

*with smooth functions  $\tilde{\alpha}_j$  and  $b$ , where  $|b| \sim 1$  and  $\tilde{\alpha}_1$  and  $\alpha_{1,1}$  are as in the Cases D, respectively ND. Moreover, in both cases we have*

$$\max\{\delta_0, \delta_3\} = \max\{\delta_0, \delta_{3,0}\}$$



*Proof.* Recall that  $\delta_0 = 2^{-k(\tilde{\kappa}_2 - 2\tilde{\kappa}_1)}$  and  $\delta_3 = 2^{-k(n_1\tilde{\kappa}_1 + \tilde{\kappa}_2 - 1)}$ . We therefore distinguish two cases:

**1. Case:**  $n_1 \neq n - 2$ . Then  $\tilde{\kappa}_2 - 2\tilde{\kappa}_1 \neq n_1\tilde{\kappa}_1 + \tilde{\kappa}_2 - 1$ , since  $\tilde{\kappa}_1 = 1/n$ , and thus either  $\delta_0 \gg \delta_3$ , or  $\delta_0 \ll \delta_3$ , for  $k$  sufficiently large. Notice that we may assume this to be true in particular if the function  $\alpha_1$  is flat, since we may then choose  $n_1$  as large as we want. By putting

$$\delta_{3,0} := \max\{\delta_0, \delta_3\},$$

we then clearly may write  $a_1$  as in Case ND.

**2. Case:**  $n_1 = n - 2$ . Then  $\delta_0 = \delta_3$ , and we may assume that  $|\alpha_1| \sim 1$ . Thus, expanding around  $x_1 = 0$  and applying (8.12), we see that we may write

$$\begin{aligned} a_1(x_1, \delta, s_2) &= \delta_0 \left( s_2 + x_1^c(\delta_1, s_2)^{n_1} \alpha_1(\delta_1(x_1^c(\delta_1, s_2))) + x_1 \alpha_{1,1}(x_1, \delta_0^r, s_2) \right) \\ &= \delta_0 s_2 \left( 1 + G_1(\delta_1 s_2^{\frac{1}{n-2}}) \alpha_1(\delta_1 s_2^{\frac{1}{n-2}} G_1(\delta_1 s_2^{\frac{1}{n-2}})) \right) + \delta_0 x_1 \alpha_{1,1}(x_1, \delta_0^r, s_2) \\ &= \delta_0 s_2 \left( 1 + g(\delta_1 s_2^{\frac{1}{n-2}}) \right) + \delta_0 x_1 \alpha_{1,1}(x_1, \delta_0^r, s_2), \end{aligned}$$

with smooth functions  $1 + g$  and  $\alpha_{1,1}$ , where  $|\alpha_{1,1}| \sim 1$ . By means of a Taylor expansion of  $g$  around the origin we thus find that

$$\begin{aligned} a_1(x_1, \delta, s_2) &= \delta_0 s_2 (\delta_1 s_2^{\frac{1}{n-2}})^N g_N(\delta_1 s_2^{\frac{1}{n-2}}) + \delta_0 x_1 \alpha_{1,1}(x_1, \delta_0^r, s_2) \\ &= (\delta_0 \delta_1^N) \tilde{\alpha}_1(\delta_0^r, s_2) + \delta_0 x_1 \alpha_{1,1}(x_1, \delta_0^r, s_2), \end{aligned}$$

with  $N \in \mathbb{N}$  and  $g_N$  smooth. Moreover, we may either assume that  $|\tilde{\alpha}_1| \sim 1$  (if  $1 + g$  is a finite type  $N$  at the origin), or that we may choose  $N$  as large as we wish (if  $1 + g$  is flat). Notice that if  $N = 0$ , then we can include the second term into the first term and arrive again at Case ND. In all other cases we arrive at the situation described by Case D, where the second term cannot be included into the first term. Notice that then  $\delta_{3,0} := \delta_0 \delta_1^N \ll \delta_0$ . Q.E.D.

Let us next introduce the quantity

$$(8.18) \quad \rho := \begin{cases} \delta_{3,0}^{\frac{B}{B-1}} + \sum_{j=2}^{B-2} \delta_{j+2}^{\frac{B}{B-j}} & \text{in Case ND,} \\ \delta_0^{\frac{3B}{2B-3}} + \delta_{3,0}^{\frac{B}{B-1}} + \sum_{j=2}^{B-2} \delta_{j+2}^{\frac{B}{B-j}} & \text{in Case D,} \end{cases}$$

which we shall view as a function  $\rho(\tilde{\delta})$  of the coefficients

$$\tilde{\delta} := \begin{cases} (\delta_{3,0}, \delta_4, \dots, \delta_B) & \text{in Case ND,} \\ (\delta_0, \delta_{3,0}, \delta_4, \dots, \delta_B) & \text{in Case D.} \end{cases}$$

**Remark 8.5.** Observe that if we scale the complete phase  $\Phi^\sharp$  from (8.10) in  $x_1$  by the factor  $r^{-1/3}$  and in  $x_2$  by  $r^{-1/B}$ ,  $r > 0$ , and multiply by  $r$ , i.e., if we look at

$$\Phi_r^\sharp(u_1, u_2, \delta, s) := r \Phi^\sharp(r^{-1/3} u_1, r^{-1/B} u_2, \delta, s),$$

then the effect is essentially that  $\tilde{\delta}$  is replaced by

$$(8.19) \quad \tilde{\delta}^r := \begin{cases} (r^{(B-1)/B} \delta_{3,0}, \dots, r^{(B-j)/B} \delta_{j+2}, \dots, r^{2/B} \delta_B) & \text{in Case ND,} \\ (r^{\frac{2B-3}{3B}} \delta_0, r^{(B-1)/B} \delta_{3,0}, \dots, r^{(B-j)/B} \delta_{j+2}, \dots, r^{2/B} \delta_B) & \text{in Case D,} \end{cases}$$

whereas  $B_1(s, \delta_1)$  is replaced by  $r^{2/3} B_1(s_2, \delta_1)$  and  $B_0(s, \delta_1)$  by  $r B_0(s, \delta_1)$ . More precisely, if denote by  $\sigma_r$  the dilations  $\sigma_r(x_1, x_2) := (r^{-\frac{1}{3}} x_1, r^{-\frac{1}{B}} x_2)$ , then we have

$$(8.20) \quad \Phi_r^\sharp(u_1, u_2, \delta, s) = u_1^3 B_3(s_2, \delta_1, r^{-\frac{1}{3}} u_1) - u_1 r^{\frac{2}{3}} B_1(s, \delta_1) + r B_0(s, \delta_1) + \phi_r^\sharp(u_1, u_2, \tilde{\delta}^r, s_2),$$

where

$$(8.21) \quad \begin{aligned} \phi_r^\sharp(u, \tilde{\delta}^r, s) &:= u_2^B b(\sigma_{r^{-1}} u, \delta_0^r, s_2) + \sum_{j=2}^{B-2} \tilde{\delta}_{j+2}^r u_2^j \tilde{\alpha}_j(r^{-\frac{1}{3}} u_1, \delta_0^r, s_2) \\ &+ \tilde{\delta}_{3,0}^r u_2 \tilde{\alpha}_1(r^{-\frac{1}{3}} u_1, \delta_0^r, s_2) + \tilde{\delta}_0^r u_1 u_2 \alpha_{1,1}(r^{-\frac{1}{3}} u_1, \delta_0^r, s_2), \end{aligned}$$

And, under these dilations,  $\rho$  is homogeneous of degree 1, i.e.,

$$(8.22) \quad \rho(\tilde{\delta}^r) = r \rho(\tilde{\delta}), \quad r > 0.$$

In particular, after scaling  $\Phi^\sharp$  in this way by  $r := 1/\rho(\tilde{\delta})$ , we see that we have normalized the coefficients of  $\phi^\sharp$  in such a way that  $\rho(\tilde{\delta}) = 1$ .

This observation, which is based on ideas by Duistermaat [11], will become important in the sequel.

**8.2. The case where  $\lambda \rho(\tilde{\delta}) \lesssim 1$ .** Assume now that  $B \in \{3, 4\}$ . Following the proof of Proposition 5.2 (c) in [21], we define the functions  $\nu_{\delta, Ai}^\lambda$  and  $\nu_{\delta, l}^\lambda$  by

$$(8.23) \quad \widehat{\nu_{\delta, Ai}^\lambda}(\xi) := \chi_0(\lambda^{\frac{2}{3}} B_1(s, \delta_1)) \widehat{\nu_\delta^\lambda}(\xi),$$

$$(8.24) \quad \widehat{\nu_{\delta, l}^\lambda}(\xi) := \chi_1((2^{-l} \lambda)^{\frac{2}{3}} B_1(s, \delta_1)) \widehat{\nu_\delta^\lambda}(\xi), \quad M_0 \leq 2^l \leq \frac{\lambda}{M_1},$$

so that

$$\nu_\delta^\lambda = \nu_{\delta, Ai}^\lambda + \sum_{M_0 \leq 2^l \leq \frac{\lambda}{M_1}} \nu_{\delta, l}^\lambda.$$

Here,  $\chi_0, \chi_1 \in C_0^\infty(\mathbb{R})$ , and  $\chi_1(t)$  is supported where  $2^{-4/3} \leq |t| \leq 2^{4/3}$ , whereas  $\chi_0(t) \equiv 1$  for  $|t| \leq M_0^{2/3}$ . Thus, by choosing  $M_0$  sufficiently large, we may assume that  $2^{-l} \leq 1/M_0 \ll 1$ . Denote by  $T_{\delta, Ai}^\lambda$  and  $T_{\delta, l}^\lambda$  the corresponding operators of convolution with the Fourier transforms of these functions.

Our goal will be to adjust the proofs of the estimates in Lemma 6.5 and Lemma 6.6 of [21] to our present situation in order to derive the following estimates, which are analogous to the corresponding ones in [21] (formally, we only have to replace a factor  $\lambda^{-1/2}$  in the estimates in [21] by the factor  $\lambda^{-1/B}$ ):

$$(8.25) \quad \|\widehat{\nu_{\delta, Ai}^\lambda}\|_\infty \leq C_1 \lambda^{-\frac{1}{B}-\frac{1}{3}}$$

$$(8.26) \quad \|\nu_{\delta, Ai}^\lambda\|_\infty \leq C_2 \lambda^{\frac{5}{3}-\frac{1}{B}},$$

as well as

$$(8.27) \quad \|\widehat{\nu_{\delta, l}^\lambda}\|_\infty \leq C_1 2^{-\frac{l}{6}} \lambda^{-\frac{1}{B}-\frac{1}{3}}$$

$$(8.28) \quad \|\nu_{\delta, l}^\lambda\|_\infty \leq C_2 2^{\frac{l}{3}} \lambda^{\frac{5}{3}-\frac{1}{B}}$$

8.2.1. *Estimates for  $\nu_{\delta, Ai}^\lambda$ .* Changing coordinates from  $x$  to  $u$  by putting  $x = \sigma_{1/\lambda} u = (\lambda^{-1/3} u_1, \lambda^{-1/B} u_2)$  in the integral (8.15), and making use of Remark 8.5 (with  $r := \lambda$ ), we find that

$$(8.29) \quad \begin{aligned} \widehat{\nu_{\delta, Ai}^\lambda}(\xi) &= \lambda^{-\frac{1}{B}-\frac{1}{3}} \chi_1(s, s_3) \chi_0(\lambda^{\frac{2}{3}} B_1(s, \delta_1)) e^{-is_3 \lambda B_0(s, \delta_1)} \\ &\times \iint e^{-is_3 \left( u_1^3 B_3(s_2, \delta_1, \lambda^{-\frac{1}{3}} u_1) - u_1 \lambda^{\frac{2}{3}} B_1(s, \delta_1) + \phi_\lambda^\sharp(u_1, u_2, \tilde{\delta}^\lambda, s_2) \right)} a(\sigma_{\lambda^{-1}} u, \delta, s) du_1 du_2, \end{aligned}$$

where  $\chi_1(s, s_3) := \chi_1(s_1 s_3) \chi_1(s_2 s_3) \chi_1(s_3)$  localizes to the region where  $s_j \sim 1, j = 1, 2, 3$ . Observe that here we are integrating over the large domain where  $|u_1| \leq \varepsilon \lambda^{1/3}$  and  $|u_2| \leq \varepsilon \lambda^{1/B}$ . Recall also that  $\phi_\lambda^\sharp$  is given by (8.21), and that  $\rho(\tilde{\delta}^\lambda) = \lambda \rho(\tilde{\delta}) \lesssim 1$ , and so we have

$$|\tilde{\delta}^\lambda| \lesssim 1 \quad \text{and} \quad \lambda^{\frac{2}{3}} |B_1(s, \delta_1)| \lesssim 1.$$

By means of this integral formula for  $\widehat{\nu_{\delta, Ai}^\lambda}(\xi)$ , we easily obtain

**Lemma 8.6.** *If  $\lambda \rho(\tilde{\delta}) \lesssim 1$ , then we may write*

$$(8.30) \quad \begin{aligned} \widehat{\nu_{\delta, Ai}^\lambda}(\xi) &= \lambda^{-\frac{1}{B}-\frac{1}{3}} \chi_1(s, s_3) \chi_0(\lambda^{\frac{2}{3}} B_1(s, \delta_1)) e^{-is_3 \lambda B_0(s, \delta_1)} \\ &\times a(\lambda^{\frac{2}{3}} B_1(s, \delta_1), \tilde{\delta}^\lambda, s, s_3, \delta_0^\sharp, \lambda^{-\frac{1}{3B}}), \end{aligned}$$

where  $a$  is again a smooth function of all its (bounded) variables.

*Proof.* We decompose the integral in (8.29) by means of suitable smooth cut-off functions into the integral  $I_1$  over the region where  $|(u_1, u_2)| \leq L$ , the integral  $I_2$  over the region where  $|(u_1, u_2)| > L$  and  $|u_2|^{B-1} \gg |u_1|$ , and the integral  $I_3$  over the region where  $|(u_1, u_2)| > L$  and  $|u_2|^{B-1} \lesssim |u_1|$ . For each of these contributions  $I_j$ , we then show that it is of the form  $a_j(\lambda^{\frac{2}{3}} B_1(s, \delta_1), \tilde{\delta}^\lambda, s, s_3, \delta_0^\sharp, \lambda^{-\frac{1}{3B}})$ , with a suitable smooth function  $a_j$ , provided  $L$  is sufficiently large. For  $I_1$ , this claim is obvious.

On the remaining region where  $|(u_1, u_2)| \geq L$ , we may use iterated integrations by parts with respect to  $u_1$ , or  $u_2$ , in order to convert the integral into an absolutely convergent integral, to which we may apply the standard rules for differentiation with

respect to parameters (such as  $s_j$ , etc.). Denote to this end by  $\Phi_c$  the complete phase function appearing in this integral. It is then easily seen that we may estimate

$$(8.31) \quad |\partial_{u_2} \Phi_c| \gtrsim |u_2|^{B-1} - c|u_1|,$$

$$(8.32) \quad |\partial_{u_1} \Phi_c| \gtrsim u_1^2 - c\lambda^{-\frac{1}{3}}|u_2|^B - c|u_2|,$$

with a fixed constant  $c > 0$ . The last terms appear only in the degenerate case D, due to the presence of the term  $\tilde{\delta}_0^\lambda u_1 u_2 \alpha_{1,1}(\lambda^{-\frac{1}{3}} u_1, \delta_0^r, s_2)$  in  $\phi_\lambda^\sharp$ .

Denote by  $I_2$  the contribution by the sub-region on which  $|u_2|^{B-1} \gg |u_1|$ . On this region, we may gain factors of order  $|u_2|^{-2N(B-1)}$  (for any  $N \in \mathbb{N}$ ) in the amplitude, by means of iterated integrations by parts in  $u_2$ . And, since  $|u_2|^{-2N(B-1)} \lesssim |u_2|^{-N}|u_2|^{-N(B-1)}$ , we see that we arrive at an absolutely convergent integral.

Similarly, denote by  $I_3$  the contribution by the sub-region on which  $|u_2|^{B-1} \lesssim |u_1|$ . Observe that  $|u_2|^B \lesssim |u_1|^{B/(B-1)} \ll u_1^2$ , since  $B \geq 3$ , and since we may assume that  $|u_1| \gg 1$ . This shows that in this region, we have  $|\partial_{u_1} \Phi_c| \gtrsim u_1^2$ , and thus we may gain factors of order  $|u_1|^{-2N}$  (for any  $N \in \mathbb{N}$ ) in the amplitude, by means of iterated integrations by parts in  $u_1$ . And, since  $|u_1|^{-2N} \lesssim |u_1|^{-N}|u_2|^{-N(B-1)}$ , we see that we arrive again at an absolutely convergent integral.

Q.E.D.

Lemma 8.5 implies in particular estimate (8.25). As for the more involved estimate (8.26), with Lemma 8.6 at hand, we can basically follow the arguments from Subsection 6.1 in [21], only with the factor  $\lambda^{-1/2}$  appearing there replaced by the factor  $\lambda^{-1/B}$  here, and with the amplitude  $g(\lambda^{2/3} B_1(s', \delta, \sigma), \lambda, \delta, \sigma, s)$  in [21] replaced by our amplitude  $a(\lambda^{\frac{2}{3}} B_1(s, \delta_1), \tilde{\delta}^\lambda, s, s_3, \delta_0^r, \lambda^{-\frac{1}{3B}})$  here (compare with (6.18) in [21]).

8.2.2. *Estimates for  $\nu_{\delta,l}^\lambda$ .* Changing coordinates from  $x$  to  $u$  by putting here  $x = \sigma_{2^l/\lambda} u$  in the integral (8.15), and making use of Remark 8.5 (with  $r := \lambda/2^l$ ), we find that

$$(8.33) \quad \begin{aligned} \widehat{\nu_{\delta,l}^\lambda}(\xi) &= (2^{-l}\lambda)^{-\frac{1}{B}-\frac{1}{3}} \chi_1(s, s_3) \chi_1((2^{-l}\lambda)^{\frac{2}{3}} B_1(s, \delta_1)) e^{-is_3 \lambda B_0(s, \delta_1)} \\ &\times \iint e^{-is_3 2^l \Phi(u_1, u_2, s, \delta, \lambda, l)} a(\sigma_{2^l/\lambda} u, \delta, s) du_1 du_2, \end{aligned}$$

with phase function

$$(8.34) \quad \begin{aligned} \Phi(u_1, u_2, s, \delta, \lambda, l) &:= \\ u_1^3 B_3(s_2, \delta_1, (2^{-l}\lambda)^{-\frac{1}{3}} u_1) &- u_1 (2^{-l}\lambda)^{\frac{2}{3}} B_1(s, \delta_1) + \phi_{2^{-l}\lambda}^\sharp(u_1, u_2, \tilde{\delta}^{2^{-l}\lambda}, s_2). \end{aligned}$$

Observe that we are here integrating over the large domain where  $|u_1| \leq (\varepsilon 2^{-l}\lambda)^{1/3}$  and  $|u_2| \leq \varepsilon (2^{-l}\lambda)^{1/B}$ . Recall also that  $\phi_{2^{-l}\lambda}^\sharp$  is given by (8.21), and that  $\rho(\tilde{\delta}^{2^{-l}\lambda}) = 2^{-l}\lambda \rho(\tilde{\delta}) \lesssim 2^{-l}$ , so that we have

$$|\tilde{\delta}^{2^{-l}\lambda}| \ll 1 \quad \text{and} \quad (2^{-l}\lambda)^{\frac{2}{3}} |B_1(s, \delta_1)| \sim 1.$$

Notice that this implies that

$$(8.35) \quad \begin{aligned} \phi_{2^{-l}\lambda}^\sharp(u_1, u_2, \tilde{\delta}^{2^{-l}\lambda}, s_2) &= u_2^B b(\sigma_{(2^{-l}\lambda)^{-1}} u, \delta_0^r, s_2) + \tilde{\delta}_0 u_1 u_2 \alpha_{1,1}((2^{-l}\lambda)^{-\frac{1}{3}} u_1, \delta_0^r, s_2) \\ &\quad + \text{small error.} \end{aligned}$$

where we have abbreviated  $\tilde{\delta}_0 := \tilde{\delta}_0^{2^{-l}\lambda}$ . Recall also that the second term appears only in the case D and that we are here assuming that  $2^{-l}\lambda \gg 1$ .

Arguing somewhat like in Section 7 of [21], we first decompose

$$\nu_{\delta,l}^\lambda = \nu_{l,0}^\lambda + \nu_{l,\infty}^\lambda,$$

where

$$\begin{aligned} \widehat{\nu_{l,0}^\lambda}(\xi) &:= (2^{-l}\lambda)^{-\frac{1}{B}-\frac{1}{3}} \chi_1(s, s_3) \chi_1((2^{-l}\lambda)^{\frac{2}{3}} B_1(s, \delta_1)) e^{-is_3 \lambda B_0(s, \delta_1)} \\ &\quad \times \iint e^{-is_3 2^l \Phi(u_1, u_2, s, \delta, \lambda, l)} a(\sigma_{2^l \lambda^{-1}} u, \delta, s) \chi_0(u) du_1 du_2, \\ \widehat{\nu_{l,\infty}^\lambda}(\xi) &:= (2^{-l}\lambda)^{-\frac{1}{B}-\frac{1}{3}} \chi_1(s, s_3) \chi_1((2^{-l}\lambda)^{\frac{2}{3}} B_1(s, \delta_1)) e^{-is_3 \lambda B_0(s, \delta_1)} \\ &\quad \times \iint e^{-is_3 2^l \Phi(u_1, u_2, s, \delta, \lambda, l)} a(\sigma_{2^l \lambda^{-1}} u, \delta, s) (1 - \chi_0(u)) du_1 du_2. \end{aligned}$$

Here, we choose  $\chi_0 \in C_0^\infty(\mathbb{R}^2)$  such that  $\chi_0(u) \equiv 1$  for  $|u| \leq L$ , where  $L$  is supposed to be a sufficiently large positive constant.

Let us first consider the contribution given by the  $\nu_{l,\infty}^\lambda$ : Arguing as in the proof of Lemma 8.6, we can easily see by means of integrations by parts that, given  $k \in \mathbb{N}$ , then for every  $N \in \mathbb{N}$ , we may write

$$(8.36) \quad \begin{aligned} \widehat{\nu_{l,\infty}^\lambda}(\xi) &= 2^{-lN} (2^{-l}\lambda)^{-\frac{1}{B}-\frac{1}{3}} \chi_1(s, s_3) \chi_1((2^{-l}\lambda)^{\frac{2}{3}} B_1(s, \delta_1)) e^{-is_3 \lambda B_0(s, \delta_1)} \\ &\quad \times a_{N,l} \left( (2^{-l}\lambda)^{\frac{2}{3}} B_1(s, \delta_1), s, s_3, \tilde{\delta}^{2^{-l}\lambda}, \delta_0^r, (2^{-l}\lambda)^{-\frac{1}{3}}, \lambda^{-\frac{1}{3B}} \right), \end{aligned}$$

where  $a_{N,l}$  is a smooth function of all its (bounded) variables such that  $\|a_{N,l}\|_{C^k}$  is uniformly bounded in  $l$ . In particular, we see that

$$(8.37) \quad \|\widehat{\nu_{l,\infty}^\lambda}\|_\infty \lesssim 2^{-lN} \lambda^{-\frac{1}{B}-\frac{1}{3}} \quad \forall N \in \mathbb{N}.$$

Next, applying the Fourier inversion formula and changing coordinates from  $s_1$  to

$$(8.38) \quad z := (2^{-l}\lambda)^{\frac{2}{3}} B_1(s, \delta_1), \quad \text{i.e.,} \quad s_1 = s_2^{\frac{n-1}{n-2}} G_3(s_2, \delta_1) - (2^{-l}\lambda)^{-\frac{2}{3}} z,$$

as in [21], we find that

$$(8.39) \quad \begin{aligned} \nu_{l,\infty}^\lambda(x) &= \lambda^3 2^{-lN} (2^{-l}\lambda)^{-\frac{1}{B}-1} \int e^{-is_3 \lambda \Psi(z, s_2, \delta)} \chi_1 \left( s_2^{\frac{n-1}{n-2}} G_3(s_2, \delta_1) - (2^{-l}\lambda)^{-\frac{2}{3}} z, s_2, s_3 \right) \\ &\quad \times \chi_1(z) a_{N,l} \left( z, s_2, s_3, \delta_0^r, \tilde{\delta}^{2^{-l}\lambda}, (2^{-l}\lambda)^{-\frac{1}{3}}, \lambda^{-\frac{1}{3B}} \right) dz ds_2 ds_3, \end{aligned}$$

where the phase function  $\Psi$  is given by

$$(8.40) \quad \begin{aligned} \Psi(z, s_2, \delta) &:= s_2^{\frac{n}{n-2}} G_5(s_2, \delta) - x_1 s_2^{\frac{n-1}{n-2}} G_3(s_2, \delta) - s_2 x_2 - x_3 \\ &\quad + (2^l \lambda^{-1})^{\frac{2}{3}} z \left( x_1 - s_2^{\frac{1}{n-2}} G_1(s_2, \delta) \right) \end{aligned}$$

(compare (7.4) in [21]). Applying our van der Corput type lemma (of order 3) to the integration in  $s_2$ , which allows for the gain of a factor  $\lambda^{-1/3}$ , this easily implies that

$$(8.41) \quad \|\nu_{l,\infty}^\lambda\|_\infty \lesssim 2^{-lN} \lambda^{\frac{5}{3}-\frac{1}{B}} \quad \forall N \in \mathbb{N}.$$

Let us next look at the contribution given by the  $\nu_{l,0}^\lambda$ : Observe first that on a region where  $|u_1|$  is sufficiently small, iterated integrations by parts with respect to  $u_1$  allow to gain factors  $2^{-lN}$  in the integral defining  $\nu_{l,0}^\lambda$ , for every  $N \in \mathbb{N}$ . Freezing afterwards  $u_1$ , we can reduce to the integration in  $u_2$  alone. A similar argument applies whenever we are allowed to integrate by parts in  $u_1$  (this is also the case when  $B_1$  and  $B_3$  have opposite signs). We shall therefore assume from now on that  $B_1$  and  $B_3$  have the same sign. Moreover, let us assume without loss of generality that  $u_1 > 0$ . Then there is a unique non-degenerate critical point  $u_1^c = u_1^c(2^{-l}\lambda)^{-1/B} u_2, \tilde{\delta}^{2^{-l}\lambda}, \dots$  of the phase  $\Phi$  in (8.34), of size  $|u_1^c| \sim 1$ , and we may restrict the integration in  $u_1$  to a small neighborhood of  $u_1^c$ . I.e., we may replace the cut-off function  $\chi_0(u_1)$  by a cut-off function  $\chi_1(u_1)$  supported in a sufficiently small neighborhood of a point  $u_1^0$  containing  $u_1^c$ . Then the method of stationary phase shows that the corresponding integral in  $u_1$  will be of order  $2^{-l/2}$ , so that this term will give the main contribution.

For the sake of simplicity, we shall therefore restrict ourselves in the sequel to the discussion of this main term  $\nu_{l,1}^\lambda$ , given by

$$(8.42) \quad \begin{aligned} \widehat{\nu_{l,1}^\lambda}(\xi) &:= (2^{-l}\lambda)^{-\frac{1}{B}-\frac{1}{3}} \chi_1(s, s_3) \chi_1((2^{-l}\lambda)^{\frac{2}{3}} B_1(s, \delta_1)) e^{-is_3 \lambda B_0(s, \delta_1)} \\ &\quad \times \iint e^{-is_3 2^l \Phi(u_1, u_2, s, \delta, \lambda, l)} a(\sigma_{2^l \lambda^{-1}} u, \delta, s) \chi_0(u) \chi_1(u_1) du_1 du_2, \end{aligned}$$

where  $|u_1| \sim 1$  on the support of  $\chi_1(u_1)$ .

Applying first the method of stationary phase to the integration in  $u_1$ , and subsequently van der Corput's estimate of order  $B$  to the integration in  $u_2$ , we easily arrive at the estimate

$$(8.43) \quad \|\widehat{\nu_{l,1}^\lambda}\|_\infty \lesssim (2^{-l}\lambda)^{-\frac{1}{B}-\frac{1}{3}} 2^{-\frac{l}{2}-\frac{l}{B}},$$

which is exactly what we need to verify (8.27) (recall here also estimate (8.37)).

As for the more involved estimation of  $\nu_{l,1}^\lambda(x)$ , Fourier inversion allows to write

$$(8.44) \quad \begin{aligned} \nu_{l,1}^\lambda(x) &= \lambda^3 (2^{-l}\lambda)^{-\frac{1}{B}-\frac{1}{3}} \int e^{-is_3 \Psi(u, s, x, \delta, \lambda, l)} \chi_1(s, s_3) \chi_1((2^{-l}\lambda)^{\frac{2}{3}} B_1(s, \delta_1)) \\ &\quad \times a(\sigma_{2^l \lambda^{-1}} u, \delta, s) \chi_0(u) \chi_1(u_1) du_1 du_2 ds \end{aligned}$$

where the complete phase  $\Psi$  is given by

$$(8.45) \quad \Psi(u, s, x, \delta, \lambda, l) := 2^l \Phi(u_1, u_2, s, \delta, \lambda, l) + \lambda \left( B_0(s, \delta_1) - s_1 x_1 - s_2 x_2 - x_3 \right),$$

with  $\Phi$  given by (8.34). Changing again from the coordinate  $s_1$  to  $z$  as in (8.38), we may also write

$$(8.46) \quad \begin{aligned} \nu_{l,1}^\lambda(x) &= \lambda^3 (2^{-l} \lambda)^{-\frac{1}{B}-1} \int e^{-is_3 \tilde{\Psi}(u, z, s_2, x, \delta, \lambda, l)} \chi_1(s, s_3) \chi_1(z) \\ &\quad \times \tilde{a}(\sigma_{2^l \lambda^{-1}} u, (2^l \lambda^{-1})^{\frac{2}{3}} z, s_2, \delta) \chi_0(u) \chi_1(u_1) du_1 du_2 dz ds_2 ds_3, \end{aligned}$$

with phase

$$(8.47) \quad \begin{aligned} \tilde{\Psi}(u, z, s_2, x, \delta, \lambda, l) &:= \lambda (2^l \lambda^{-1})^{\frac{2}{3}} z \left( x_1 - s_2^{\frac{1}{n-2}} G_1(s_2, \delta) \right) \\ &\quad + \lambda \left( s_2^{\frac{n}{n-2}} G_5(s_2, \delta) - x_1 s_2^{\frac{n-1}{n-2}} G_3(s_2, \delta) - s_2 x_2 - x_3 \right) \\ &\quad + 2^l \left( u_1^3 B_3(s_2, \delta_1, (2^{-l} \lambda)^{-\frac{1}{3}} u_1) - z u_1 + \phi_{2^{-l} \lambda}^\sharp(u_1, u_2, \tilde{\delta}^{2^{-l} \lambda}, s_2) \right). \end{aligned}$$

We shall prove the following estimate:

$$(8.48) \quad |\nu_{l,1}^\lambda(x)| \leq C 2^{\frac{l}{3}} \lambda^{\frac{5}{3}-\frac{1}{B}},$$

with a constant  $C$  which is independent of  $\lambda, l, x$  and  $\delta$ .

As in [21], this estimate is easily verified if  $|x| \gg 1$ , simply by means of integrations by parts in the variables  $s_2, s_3$  and  $z$ , in combination with the method of stationary phase in  $u_1$  and van der Corput's estimate of order  $B$  in  $u_2$ . Similarly, if  $|x| \lesssim 1$  and  $|x_1| \ll 1$ , we can arrive at the same conclusion, by first integrating by parts in  $z$ . Indeed, in these situations we may gain factors  $2^{-2Nl/3} \lambda^{-N/3}$  through integrations by parts, so that the corresponding estimates can be summed in a trivial way.

Let us thus assume that  $|x| \lesssim 1$  and  $|x_1| \sim 1$ . Following Section 7 in [21], we then decompose

$$\nu_{l,1}^\lambda = \nu_{l,I}^\lambda + \nu_{l,II}^\lambda + \nu_{l,III}^\lambda.$$

where  $\nu_{l,I}^\lambda, \nu_{l,II}^\lambda$  and  $\nu_{l,III}^\lambda$  correspond to the contribution to the integral in (8.46) given by the regions where  $\lambda (2^l \lambda^{-1})^{\frac{2}{3}} |x_1 - s_2^{\frac{1}{n-2}} G_1(s_2, \delta)| \gg 2^l$ ,  $\lambda (2^l \lambda^{-1})^{\frac{2}{3}} |x_1 - s_2^{\frac{1}{n-2}} G_1(s_2, \delta)| \sim 2^l$  and  $\lambda (2^l \lambda^{-1})^{\frac{2}{3}} |x_1 - s_2^{\frac{1}{n-2}} G_1(s_2, \delta)| \ll 2^l$ , respectively. The first and the last term can easily be handled as in [21] by means of integrations by parts in  $z$ , with subsequent exploitation of the oscillations with respect to  $u_1$  and  $u_2$ , which leads to the following estimate:

$$(8.49) \quad |\nu_{l,I}^\lambda(x)| + |\nu_{l,III}^\lambda(x)| \leq C 2^{-\frac{l}{3}} \lambda^{\frac{5}{3}-\frac{1}{B}},$$

which better than (8.48) by a factor  $2^{-2l/3}$ , so that summation in  $l$  is no problem for these terms. Nevertheless, summation in  $\lambda$  still will require an interpolation argument if  $B = 3$ .

Let us next concentrate on  $\nu_{l,II}^\lambda(x)$ , which is of the form

$$(8.50) \quad \begin{aligned} \nu_{l,II}^\lambda(x) &= \lambda^3 (2^{-l}\lambda)^{-\frac{1}{B}-1} \int e^{-is_3 \tilde{\Psi}(u,z,s_2,x,\delta,\lambda,l)} \tilde{a}(\sigma_{2^l\lambda^{-1}}u, (2^l\lambda^{-1})^{\frac{2}{3}}z, s_2, \delta) \chi_1(s_2, s_3) \\ &\times \chi_1\left((2^l\lambda^{-1})^{-\frac{1}{3}}(x_1 - s_2^{\frac{1}{n-2}}G_1(s_2, \delta))\right) \chi_0(u) \chi_1(u_1) \chi_1(z) du_1 du_2 dz ds_2 ds_3. \end{aligned}$$

Writing

$$\begin{aligned} \tilde{\Psi}(u, z, s_2, x, \delta, \lambda, l) &= \lambda \left( s_2^{\frac{n}{n-2}} G_5(s_2, \delta) - x_1 s_2^{\frac{n-1}{n-2}} G_3(s_2, \delta) - s_2 x_2 - x_3 \right) \\ &+ 2^l \left[ z (2^{-l}\lambda)^{\frac{1}{3}} (x_1 - s_2^{\frac{1}{n-2}} G_1(s_2, \delta)) - z u_1 + u_1^3 B_3(s_2, \delta_1, (2^{-l}\lambda)^{-\frac{1}{3}} u_1) \right. \\ &\quad \left. + \phi_{2^{-l}\lambda}^\sharp(u_1, u_2, \tilde{\delta}^{2^{-l}\lambda}, s_2) \right], \end{aligned}$$

and observing that here  $|(2^{-l}\lambda)^{\frac{1}{3}}(x_1 - s_2^{\frac{1}{n-2}}G_1(s_2, \delta))| \sim 1$  and  $|u_1| \sim 1$ , we see that the phase  $\tilde{\Psi}$  may have a critical point  $(u_1^c, z^c)$  within the support of the amplitude, as a function of  $u_1$  and  $z$ . Moreover, in a similar way as in [21], we see that this critical point will be non-degenerate. Of course, if there is no critical point, we may obtain even better estimates by means of integrations by parts. Let us thus in the sequel assume that there is such a critical point.

Applying then the method of stationary phase to the integration in  $(u_1, z)$ , we see that we essentially may write

$$\begin{aligned} \nu_{l,II}^\lambda(x) &= \lambda^2 (2^{-l}\lambda)^{-\frac{1}{B}} \int e^{-is_3 \Psi_2(u_2, s_2, x, \delta, \lambda, l)} a_2((2^l\lambda^{-1})^{\frac{1}{3}}u_2, s_2, x, (2^l\lambda^{-1})^{\frac{1}{3}}\delta) \chi_1(s_2, s_3) \\ &\times \chi_1\left((2^l\lambda^{-1})^{-\frac{1}{3}}(x_1 - s_2^{\frac{1}{n-2}}G_1(s_2, \delta))\right) \chi_0(u_2) du_2 ds_2 ds_3, \end{aligned}$$

where the phase  $\Psi_2$  arises from  $\tilde{\Psi}$  by replacing  $(u_1, z)$  by the critical point  $(u_1^c, z^c)$  (which of course also depends on the other variables).

In order to compute  $\Psi_2$  more explicitly, we go back to our original coordinates, in which our complete phase is given by (compare (8.7))

$$(8.51) \quad \lambda \left( s_1 y_1 + s_2 y_1^2 \omega(\delta_1 y_1) + y_1^n \alpha(\delta_1 y_1) + s_2 \delta_0 y_2 + y_2^B b(y_1, y_2, \delta) + r(y_1, y_2, \delta) - s_1 x_1 - s_2 x_2 - x_3 \right).$$

Recall also that we passed from the coordinates  $(y_1, s_1)$  to the coordinates  $(u_1, z)$  by means of a smooth change of coordinates (depending on the remaining variables  $(y_2, s_2)$ ). Since the value of a function at a critical point does not depend on the chosen coordinates, arguing by means of Lemma 7.1 in [21], we find that in the coordinates  $(y_2, s_2)$  the phase  $\Psi_2$  is given by

$$(8.52) \quad \lambda \left( s_2 x_1^2 \omega(\delta_1 x_1) + x_1^n \alpha(\delta_1 x_1) + s_2 \delta_0 y_2 + y_2^B b(x_1, y_2, \delta) + r(x_1, y_2, \delta) - s_2 x_2 - x_3 \right).$$



And, since  $y_2 = (2^{-l}\lambda)^{-1/B}u_2$ , this means that

$$\begin{aligned} \Psi_2(u_2, s_2, x, \delta, \lambda, l) &= \lambda \left( s_2 x_1^2 \omega(\delta_1 x_1) + x_1^n \alpha(\delta_1 x_1) - s_2 x_2 - x_3 \right) \\ &+ 2^l \left( u_2^B b(x_1, (2^{-l}\lambda)^{-\frac{1}{B}} u_2, \delta) + \sum_{j=2}^{B-3} u_2^j \tilde{\delta}_{j+2}^{2^{-l}\lambda} x_1^{n_j} \alpha_j(\delta_1 x_1) \right. \\ &\quad \left. + (2^{-l}\lambda)^{\frac{B-1}{B}} u_2 [\delta_0 s_2 + \delta_3 x_1^{n_1} \alpha_1(\delta_1 x_1)] \right) \end{aligned}$$

(compare (8.4)). Note that  $\partial_{s_2}(s_2^{\frac{1}{n-2}} G_1(s_2, \delta)) \sim 1$  because  $s_2 \sim 1$  and  $G_1(s_2, 0) = 1$ . Therefore, the relation  $|x_1 - s_2^{\frac{1}{n-2}} G_1(s_2, \delta)| \sim (2^l \lambda^{-1})^{\frac{1}{3}}$  can be re-written as  $|s_2 - \tilde{G}_1(x_1, \delta)| \sim (2^l \lambda^{-1})^{\frac{1}{3}}$ , where  $\tilde{G}_1$  is again a smooth function such that  $|\tilde{G}_1| \sim 1$ . If we write

$$s_2 = (2^l \lambda^{-1})^{\frac{1}{3}} v + \tilde{G}_1(x_1, \delta),$$

then this means that  $|v| \sim 1$ . We shall therefore change variables from  $s_2$  to  $v$ , which leads to

$$\begin{aligned} \nu_{l,II}^\lambda(x) &= \lambda^2 (2^{-l}\lambda)^{-\frac{1}{B}-\frac{1}{3}} \int e^{-is_3 \Psi_3(u_2, v, x, \delta, \lambda, l)} a_3((2^l \lambda^{-1})^{\frac{1}{3}} u_2, v, x, (2^l \lambda^{-1})^{\frac{1}{3}}, \delta) \\ (8.53) \quad &\quad \times \chi_1(s_3) \chi_1(v) \chi_0(u_2) du_2 dv ds_3, \end{aligned}$$

with a smooth amplitude  $a_3$  and the new phase function

$$\begin{aligned} \Psi_3(u_2, v, x, \delta, \lambda, l) &= \lambda \left( v (2^{-l}\lambda)^{-\frac{1}{3}} (x_1^2 \omega(\delta_1 x_1) - x_2) + (2^{-l}\lambda)^{-\frac{1}{B}-\frac{1}{3}} \delta_0 v u_2 + Q_A(x, \delta) \right) \\ (8.54) \quad &+ 2^l \left( u_2^B b(x_1, (2^{-l}\lambda)^{-\frac{1}{B}} u_2, \delta) + \sum_{j=2}^{B-3} u_2^j \tilde{\delta}_{j+2}^{2^{-l}\lambda} x_1^{n_j} \alpha_j(\delta_1 x_1) \right. \\ &\quad \left. + u_2 (2^{-l}\lambda)^{\frac{B-1}{B}} [\delta_0 \tilde{G}_1(x_1, \delta) + \delta_3 x_1^{n_1} \alpha_1(\delta_1 x_1)] \right), \end{aligned}$$

where

$$Q_A(x, \delta) := \tilde{G}_1(x_1, \delta) (x_1^2 \omega(\delta_1 x_1) - x_2) + x_1^n \alpha(\delta_1 x_1) - x_3.$$

Applying van der Corput's estimate of order  $B$  to the integration in  $u_2$  in (8.53), we find that

$$|\nu_{l,II}^\lambda(x)| \leq C \lambda^2 (2^{-l}\lambda)^{-\frac{1}{B}-\frac{1}{3}} 2^{-\frac{l}{B}} = 2^{\frac{l}{3}} \lambda^{\frac{5}{3}-\frac{1}{B}}.$$

This proves (8.48), which completes the proof of estimate (8.28).

**8.2.3. Consequences of the estimates (8.25) - (8.28).** By interpolation, these estimates imply

$$(8.55) \quad \|T_{\delta, Ai}^\lambda\|_{p_c \rightarrow p'_c} \lesssim \lambda^{-\frac{1}{3}-\frac{1}{B}+2\theta_c},$$

$$(8.56) \quad \|T_{\delta, l}^\lambda\|_{p_c \rightarrow p'_c} \lesssim 2^{-\frac{l(1-3\theta_c)}{6}} \lambda^{-\frac{1}{3}-\frac{1}{B}+2\theta_c}.$$

But, by Lemma 3.1, we have  $\theta_c \leq \tilde{\theta}_B = 3/(2B + 3)$ , and this easily implies that the exponents of  $\lambda$  and  $2^l$  in the preceding estimates is strictly negative, if  $B = 4$ , and zero, if  $B = 3$  (where  $\theta_c = 1/3$ ). We can therefore sum these estimates over all  $l$ , as well as  $\lambda \gg 1$ , if  $B = 4$ , and the desired estimate, whereas if  $B = 3$ , then we only get a uniform estimates

$$\|T_{\delta, Ai}^\lambda\|_{p_c \rightarrow p'_c} \leq C, \quad \|T_{\delta, l}^\lambda\|_{p_c \rightarrow p'_c} \leq C, \quad (\lambda \rho(\tilde{\delta}) \lesssim 1),$$

with a constant  $C$  not depending on  $\delta$ . The case where  $B = 3$  will therefore again require a complex interpolation argument in order to capture the endpoint, as in the proof of Proposition 5.2 (c) in [21].

In particular, we have proven

**Proposition 8.7.** *If  $B = 4$ , then under the assumptions in this section*

$$\sum_{\{\lambda \geq 1: \lambda \rho(\tilde{\delta}) \lesssim 1\}} \|T_\delta^\lambda\|_{p_c \rightarrow p'_c} \lesssim 1,$$

where the constant in this estimate will not depend on  $\delta$ .

In combination with the following proposition, which will be proved in the next two sections, this will complete the discussion of the remaining case where  $B = 4$  in (6.20), hence also the proof of Proposition 2.1 for this case.

**Proposition 8.8.** *If  $B = 4$ , then under the assumptions in this section*

$$\sum_{\{\lambda \geq 1: \lambda \rho(\tilde{\delta}) \gg 1\}} \|T_\delta^\lambda\|_{p_c \rightarrow p'_c} \lesssim 1,$$

where the constant in this estimate will not depend on  $\delta$ .

## 9. THE CASE WHERE $m = 2$ , $B = 4$ AND $A = 1$

According to Proposition 8.7, we are left with controlling the operators  $T_\delta^\lambda$  with  $\lambda \rho \gg 1$ , where we have used the abbreviation  $\rho = \rho(\tilde{\delta})$ . If  $A = 1$ , then according to (6.21), we have  $h^r + 1 = 4$ , hence  $\theta_c = 1/4$  and  $p'_c = 8$ . We shall then use the first estimate in (6.3), i.e.,

$$(9.1) \quad \|\nu_\delta^\lambda\|_\infty \lesssim \lambda^{\frac{7}{4}}.$$

The crucial observation is that under the assumption  $\lambda \rho \gg 1$ , we can here improve on the estimate (6.2) for  $\widehat{\nu_\delta^\lambda}$ . Indeed, we shall prove that

$$(9.2) \quad \|\widehat{\nu_\delta^\lambda}\|_\infty \lesssim \rho^{-\frac{1}{12}} \lambda^{-\frac{2}{3}}.$$

Under the assumption that this estimate is valid, we obtain from (9.1) and (9.2) by interpolation that

$$\|T_\delta^\lambda\|_{p_c \rightarrow p'_c} \lesssim (\lambda \rho)^{-\frac{1}{16}},$$

hence the desired remaining estimate

$$\sum_{\lambda\rho\gg 1} \|T_\delta^\lambda\|_{p_c\rightarrow p'_c} \lesssim 1.$$

In order to prove (9.2), recall that from (8.15) that

$$\widehat{\nu}_\delta(\xi) = \iint e^{-i\lambda s_3 \Phi^\#(x, \delta, s)} a(x, \delta, s) dx_2 dx_1,$$

where the complete phase  $\Phi^\#$  is given by

$$\Phi^\#(x, \delta, s) = x_1^3 B_3(s_2, \delta_1, x_1) - x_1 B_1(s, \delta_1) + B_0(s, \delta_1) + \phi^\#(x, \delta, s_2),$$

with

$$(9.3) \quad \begin{aligned} \phi^\#(x, \delta, s_2) = & x_2^4 b(x, \delta_0^r, s_2) + \delta_4 x_2^2 \tilde{\alpha}_2(x_1, \delta_0^r, s_2) + \delta_{3,0} x_2 \tilde{\alpha}_1(x_1, \delta_0^r, s_2) \\ & \delta_0 x_1 x_2 \alpha_{1,1}(x_1, \delta_0^r, s_2), \end{aligned}$$

and where  $a$  is a smooth amplitude supported in a small neighborhood of the origin in  $x$ . Recall also from Lemma 8.4 that  $|\alpha_{1,1}| \equiv 0$  and  $|\tilde{\alpha}_1| \sim 1$  in Case ND, whereas  $|\alpha_{1,1}| \sim 1$  and  $\tilde{\alpha}_1$  is independent of  $x_1$  in Case D. Recall also that  $s_j \sim 1$  for  $j = 1, 2, 3$ .

Moreover, in Case ND we have

$$\rho = \delta_{3,0}^{\frac{4}{3}} + \delta_4^2,$$

whereas in Case D

$$\rho = \delta_0^{\frac{12}{5}} + \delta_{3,0}^{\frac{4}{3}} + \delta_4^2,$$

where  $\delta_{3,0} \ll \delta_0$ . We shall treat both cases ND and D at the same time, assuming implicitly that  $\delta_0 = 0$  in Case ND.

Estimate (9.2) will thus be a direct consequence of the following lemma, which can be derived from more general results by Duistermaat (cf. Proposition 4.3.1 in [11]). For the convenience of the reader, we shall give a more elementary, direct proof for our situation, which requires only  $C^2$ -smoothness of the amplitude. Our approach will be based on arguments similar to the ones used on pp.196- 205 in [19].

**Lemma 9.1.** *Denote by  $J(\lambda, \delta, s)$  the oscillatory*

$$J(\lambda, \delta, s) = \chi_1(s_1, s_2) \iint e^{-i\lambda \Phi(x, \delta, s)} a(x, \delta, s) dx_1 dx_2,$$

*with phase*

$$\Phi(x, \delta, s) = x_1^3 B_3(s_2, \delta_1, x_1) - x_1 B_1(s, \delta_1) + \phi^\#(x_1, x_2, \delta, s_2),$$

*where  $\phi^\#$  is given by (9.3), and where  $\chi_1(s_1, s_2)$  localizes to the region where  $s_j \sim 1, j = 1, 2$ . Let us also put*

$$\tilde{\rho} := \rho + |B_1(s, \delta_1)|^{\frac{3}{2}}.$$

Then the following estimate

$$(9.4) \quad |J(\lambda, \delta, s)| \leq \frac{C}{\tilde{\rho}^{\frac{1}{12}} \lambda^{\frac{2}{3}}}$$

holds true, provided the amplitude  $a$  is supported in a sufficiently small neighborhood of the origin. The constant  $C$  in this estimate is independent of  $\delta$  and  $s$ .

*Proof.* Note that  $\tilde{\rho} \lesssim 1$ . We may indeed even assume that  $\tilde{\rho} \ll 1$ .

For, if  $|B_1(s, \delta_1)| \sim 1$ , and if we choose the support of  $a$  sufficiently small in  $x$ , then it is easily seen that the phase  $\Phi$  has no critical point with respect to  $x_1$  on the support of the amplitude, and thus an integration by parts in  $x_1$  allows to estimate  $|J(\lambda, \delta, s)| \leq C\lambda^{-1}$ , which is better than what is needed for (9.4). And, if  $|B_1(s, \delta_1)| \ll 1$ , then we also have  $\tilde{\rho} \ll 1$ .

We begin with the case where  $\tilde{\rho}\lambda \lesssim 1$ . Here we can argue as in the proof of Lemma 8.6: Changing coordinates from  $x$  to  $u$  by putting  $x = \sigma_{1/\lambda}u = (\lambda^{-1/3}u_1, \lambda^{-1/4}u_2)$  and making use of Remark 8.5 (with  $r := \lambda$ ), we find that

$$\begin{aligned} J(\lambda, \delta, s) &= \lambda^{-\frac{1}{4}-\frac{1}{3}} \chi_1(s_1, s_2) \\ &\quad \times \iint e^{-is_3 \left( u_1^3 B_3(s_2, \delta_1, \lambda^{-\frac{1}{3}}u_1) - u_1 \lambda^{\frac{2}{3}} B_1(s, \delta_1) + \phi_\lambda^\sharp(u_1, u_2, \tilde{\delta}^\lambda, s_2) \right)} a(\sigma_{\lambda^{-1}}u, \delta, s) du_1 du_2. \end{aligned}$$

Observe that here we are integrating over the large domain where  $|u_1| \leq \varepsilon\lambda^{1/3}$  and  $|u_2| \leq \varepsilon\lambda^{1/4}$ . Recall also that  $\phi_\lambda^\sharp$  is given by (8.21), and that  $\rho(\tilde{\delta}^\lambda) = \lambda\rho(\tilde{\delta}) \lesssim 1$ , and so we have

$$|\tilde{\delta}^\lambda| \lesssim 1 \quad \text{and} \quad \lambda^{\frac{2}{3}}|B_1(s, \delta_1)| \lesssim 1.$$

It is then easily seen by means of integrations by parts in  $u_1$  respectively  $u_2$  (when-ever these quantities are large) that the double-integral in this expression is uniformly bounded in  $\delta$  and  $s$ , and thus we arrive at the uniform estimate

$$|J(\lambda, \delta, s)| \leq \frac{C}{\lambda^{\frac{7}{12}}}.$$

This estimate is stronger than estimate (9.4) when  $\tilde{\rho}\lambda \lesssim 1$ .

From now on, we may thus assume that  $\Lambda := \tilde{\rho}\lambda \gtrsim 1$ . We then apply the change of coordinates  $x = \sigma_{\tilde{\rho}}u = (\tilde{\rho}^{1/3}u_1, \tilde{\rho}^{1/4}u_2)$  and find that

$$J(\lambda, \delta, s) = \tilde{\rho}^{\frac{7}{12}} I(\lambda\tilde{\rho}, \delta, s),$$

where we have put

$$(9.5) \quad I(\Lambda, \delta, s) = \chi_1(s_1, s_2) \iint e^{-i\Lambda\Phi_1(u, \delta, s)} a(\sigma_{\tilde{\rho}}u, \delta, s) du_1 du_2, \quad (\Lambda \gtrsim 1),$$

with

$$\Phi_1(u, \delta, s) := u_1^3 B_3(s_2, \delta_1, \tilde{\rho}^{\frac{1}{3}}u_1) - u_1 B_1'(s, \delta_1) + \phi(u, \tilde{\rho}, \delta, s_2)$$

and

$$(9.6) \quad \begin{aligned} \phi(u, \tilde{\rho}, \delta, s_2) &:= u_2^4 b(\sigma_{\tilde{\rho}} u, \delta_0^r, s_2) + \delta_4' u_2^2 \tilde{\alpha}_2(\tilde{\rho}^{\frac{1}{3}} u_1, \delta_0^r, s_2) + \delta_{3,0}' u_2 \tilde{\alpha}_1(\tilde{\rho}^{\frac{1}{3}} u_1, \delta_0^r, s_2) \\ &\quad \delta_0' u_1 u_2 \alpha_{1,1}(\tilde{\rho}^{\frac{1}{3}} u_1, \delta_0^r, s_2). \end{aligned}$$

Here,

$$B_1'(s, \delta_1) := \frac{B_1(s, \delta_1)}{\tilde{\rho}^{\frac{2}{3}}}, \quad \delta_0' := \frac{\delta_0}{\tilde{\rho}^{\frac{5}{12}}}, \quad \delta_{3,0}' := \frac{\delta_{3,0}}{\tilde{\rho}^{\frac{3}{4}}}, \quad \delta_4' := \frac{\delta_4}{\tilde{\rho}^{\frac{1}{2}}},$$

so that, in analogy with Remark 8.5, we have

$$(9.7) \quad (\delta_0')^{\frac{12}{5}} + (\delta_{3,0}')^{\frac{4}{3}} + (\delta_4')^2 + |B_1'(s, \delta_1)|^{\frac{3}{2}} = 1.$$

Note that in particular

$$\delta_0' + \delta_{3,0}' + \delta_4' + |B_1'(s, \delta_1)| \sim 1.$$

In order to prove (9.4), we have thus to verify the following estimate:

$$(9.8) \quad |I(\Lambda, \delta, s)| \leq C \Lambda^{-\frac{2}{3}}.$$

Take again a smooth cut-off function  $\chi_0 \in C_0^\infty(\mathbb{R}^2)$  such that  $\chi_0(u) = 1$  for  $|u| \leq L$  where  $L$  is a sufficiently big fixed positive number, and decompose

$$I(\Lambda, \delta, s) = I_0(\Lambda, \delta, s) + I_\infty(\Lambda, \delta, s),$$

with

$$I_0(\Lambda, \delta, s) := \chi_1(s_1, s_2) \iint e^{-i\Lambda\Phi_1(u, \delta, s)} a(\sigma_{\tilde{\rho}} u, \delta, s) \chi_0(u) du_1 du_2,$$

and

$$I_\infty(\Lambda, \delta, s) := \chi_1(s_1, s_2) \iint e^{-i\Lambda\Phi_1(u, \delta, s)} a(\sigma_{\tilde{\rho}} u, \delta, s) (1 - \chi_0(u)) du_1 du_2.$$

Note that on the support of  $1 - \chi_0$  we have  $|u_1| \gtrsim L$  or  $|u_2| \gtrsim L$ . Thus, by choosing  $L$  sufficiently large, we see by (9.7) that the phase  $\Phi_1$  has no critical point on the support of  $1 - \chi_0$ , and in fact we may use integrations by parts in  $u_1$  respectively  $u_2$  in order to prove that the double-integral in the expression for  $I_\infty(\Lambda, \delta, s)$  is of order  $O(\Lambda^{-1})$ , uniformly in  $\delta$  and  $s$ . This is stronger than what is required for (9.8).

There remains the integral  $I_0(\Lambda, \delta, s)$ . Here we use arguments from [19] (compare pp. 203-205). Recall from (9.7) that  $(B_1'(s, \delta_1), \delta_0', \delta_{3,0}', \delta_4')$  lies on the “unite sphere”

$$\Sigma := \{(B_1', \delta_0', \delta_{3,0}', \delta_4') \in \mathbb{R}^4 : |B_1'|^{\frac{3}{2}} + (\delta_0')^{\frac{12}{5}} + (\delta_{3,0}')^{\frac{4}{3}} + (\delta_4')^2 = 1\}.$$

Following [19] let us fix a point  $((B_1')^0, (\delta_0')^0, (\delta_{3,0}')^0, (\delta_4')^0) \in \Sigma$ , a point  $s^0$  on the support of  $\chi_1$  and a point  $u^0 = (u_1^0, u_2^0) \in \text{supp } \chi_0$ , and denote by  $\eta$  a smooth cut-off function supported near  $u^0$ . By  $I_0^\eta$  we denote the corresponding oscillatory integral containing  $\eta$  as a factor in the amplitude:

$$I_0^\eta(\Lambda, \delta, s) := \chi_1(s_1, s_2) \iint e^{-i\Lambda\Phi_2(u, B_1', \delta_0', \delta_{3,0}', \delta_4', \tilde{\rho}, s)} a(\sigma_{\tilde{\rho}} u, \delta, s) \chi_0(u) \eta(u) du_1 du_2,$$

where

$$(9.9) \quad \Phi_2(u, B'_1, \delta'_0, \delta'_{3,0}, \delta'_4, \tilde{\rho}, s) := u_1^3 B_3(s_2, \delta_1, \tilde{\rho}^{\frac{1}{3}} u_1) - u_1 B'_1 + \phi(u, \tilde{\rho}, \delta, s_2),$$

with  $\phi$  as before.

We shall prove that  $I_0^\eta$  satisfies the estimate

$$(9.10) \quad |I_0^\eta(\Lambda, \delta, s)| \leq C \|a(\cdot, \delta, s)\|_{C^2} \Lambda^{-\frac{2}{3}}.$$

provided  $\eta$  is supported in a sufficiently small neighborhood of  $U$  of  $u^0$ ,  $s$  lies in a sufficiently small neighborhood  $S$  of  $s^0$  and  $(B'_1, \delta'_0, \delta'_{3,0}, \delta'_4)$  in a sufficiently small neighborhood  $V$  of the point  $((B'_1)^0, (\delta'_0)^0, (\delta'_{3,0})^0, (\delta'_4)^0)$  in  $\Sigma$ . The constant  $C$  in these estimates may depend on the “base points”  $u^0, s^0$  and  $((B'_1)^0, (\delta'_0)^0, (\delta'_{3,0})^0, (\delta'_4)^0)$ , as well as on the chosen neighborhoods, but not on  $\Lambda, \delta$  and  $s$ .

By means of a partition of the identity argument this will imply the same type estimate for  $I_0$ , hence for  $I$ , which will conclude the proof of Lemma 9.1.

Now, if  $\nabla_u \Phi_2(u^0, (B'_1)^0, (\delta'_0)^0, (\delta'_{3,0})^0, (\delta'_4)^0, \tilde{\rho}, s^0) \neq 0$ , then by using an integration by parts argument in a similar way as for  $I_\infty$  we arrive at the same estimate for  $I_0^\eta$  as for  $I_\infty$ , which is better than what is required.

We may therefore assume from now on that  $\nabla_u \Phi_2(u^0, (B'_1)^0, (\delta'_0)^0, (\delta'_{3,0})^0, (\delta'_4)^0, \tilde{\rho}, s^0) = 0$ , and shall distinguish two cases.

**Case 1:**  $u_1^0 \neq 0$ . In this case, it is easy to see from (9.9) and (9.6) that

$$\partial_{u_1}^2 \Phi_2(u^0, (B'_1)^0, (\delta'_0)^0, (\delta'_{3,0})^0, (\delta'_4)^0, \tilde{\rho}, s^0) \neq 0,$$

provided  $\tilde{\rho}$  is sufficiently small. Then, by the implicit function theorem, the phase  $\Phi_2$  has a unique critical point  $u_1^c(u_2, B'_1, \delta'_0, \delta'_{3,0}, \delta'_4, \tilde{\rho}, s)$  with respect to  $x_1$ , which is a smooth function of its variables, provided we choose the neighborhoods  $U$  etc. sufficiently small. Indeed, and when  $\tilde{\rho} = 0$ , then by (9.9) and (9.6),

$$(9.11) \quad u_1^c(u_2, B'_1, \delta'_0, \delta'_{3,0}, \delta'_4, 0, s) = \left( \frac{B'_1 - \delta'_0 u_2 \alpha_{11}(0, \delta_0^*, s_2)}{3B_3(s_2, \delta_1, 0)} \right)^{\frac{1}{2}}.$$

We may thus apply the method of stationary phase to the integration with respect to the variable  $u_1$  in the integral defining  $I_0^\eta$ . Let us denote by  $\Psi$  the phase function

$$\Psi(u_2, B'_1, \delta'_0, \delta'_{3,0}, \delta'_4, \tilde{\rho}, s) := \Phi_2(u_1^c(u_2, B'_1, \delta'_0, \delta'_{3,0}, \delta'_4, \tilde{\rho}, s), B'_1, \delta'_0, \delta'_{3,0}, \delta'_4, \tilde{\rho}, s)$$

which arises through this application of the method of stationary phase. We claim that we then have

$$(9.12) \quad \max_{j=4,5} |\partial_{u_2}^j \Psi(u_2^0, B'_1, \delta'_0, \delta'_{3,0}, \delta'_4, \tilde{\rho}, s)| \neq 0.$$

Notice that it suffices to prove this for  $\tilde{\rho} = 0$ , since then the results follows also for  $\tilde{\rho}$  sufficiently small.

In order to prove (9.12) when  $\tilde{\rho} = 0$ , we make use of (9.11). Since  $|B_3| \sim 1$ , (9.11) shows that we may assume that

$$(9.13) \quad |B'_1 - \delta'_0 u_2 \alpha_{11}(0, \delta_0^*, s_2)| \sim |u_1^c| \sim |u_1^0|.$$

Note also that by (9.11) we have

$$\Psi(u_2, B'_1, \delta'_0, \delta'_{3,0}, \delta'_4, 0, s) = \Gamma(u_2) + u_2^4 b(0, \delta_0^r, s_2) + \delta'_4 u_2^2 \tilde{\alpha}_2(0, \delta_0^r, s_2) + \delta'_{3,0} u_2 \tilde{\alpha}_1(0, \delta_0^r, s_2),$$

where we have put

$$\Gamma(u_2) := -2 \cdot 3^{-\frac{3}{2}} B_3(s_2, \delta_1, 0)^{-\frac{1}{2}} (B'_1 - \delta'_0 u_2 \alpha_{11}(0, \delta_0^r, s_2))^{\frac{3}{2}}.$$

In the case ND we have  $\alpha_{11} \equiv 0$ , and thus  $|\partial_{u_2}^4 \Psi(u_2^0, B'_1, \delta'_0, \delta'_{3,0}, \delta'_4, \tilde{\rho}, s)| \neq 0$ .

Next, if we are in Case D, then  $|\alpha_{11}| \sim 1$ , and (9.13) implies that  $|\Gamma^{(j)}(u_2)| \sim |u_1^0|^{3/2} |\delta'_0 / u_1^0|^j$ . Therefore, if  $\delta'_0 \ll |u_1^0|$ , then we find that  $|\partial_{u_2}^4 \Psi(u_2^0, B'_1, \delta'_0, \delta'_{3,0}, \delta'_4, \tilde{\rho}, s)| \neq 0$ , and if  $\delta'_0 \gtrsim |u_1^0|$ , then

$$|\partial_{u_2}^5 \Psi(u_2^0, B'_1, \delta'_0, \delta'_{3,0}, \delta'_4, \tilde{\rho}, s)| \gtrsim |u_1^0|^{\frac{3}{2}} \neq 0.$$

This verifies (9.12) also in this case. But, (9.12) allows to apply van der Corput's lemma to the integration in  $u_2$  (after the application of the method of stationary phase in  $u_1$ ) and thus altogether obtain the estimate

$$|I_0^\eta(\Lambda, \delta, s)| \leq C \Lambda^{-\frac{1}{2} - \frac{1}{5}},$$

which is again even stronger than what is required by (9.10).

**Case 2:**  $u_1^0 = 0$ . Assume first,  $(\delta'_0)^0 \neq 0$  and  $|\alpha_{11}| \sim 1$  (this situation can occur only in the case D). Then

$$\begin{aligned} \partial_{u_1} \partial_{u_2} \Phi_2(u^0, (B'_1)^0, (\delta'_0)^0, (\delta'_{3,0})^0, (\delta'_4)^0, 0, s^0) &= \delta'_0 \alpha_{11}(0, 0, s_2^0) \neq 0, \\ \partial_{u_1}^2 \Phi_2(u^0, (B'_1)^0, (\delta'_0)^0, (\delta'_{3,0})^0, (\delta'_4)^0, 0, s^0) &= 0. \end{aligned}$$

Therefore we can apply the method of stationary phase to the integration in both variables  $(u_1, u_2)$  and again obtain an estimate of order  $O(\Lambda^{-1})$ , which is again stronger than what we need.

From now on, we may thus assume that  $(\delta'_0)^0 = 0$  (recall that in Case ND, we have  $\alpha_{11} \equiv 0$  and are assuming that  $\delta_0 = 0$ , hence also  $\delta'_0 = 0$ , so that this assumption is automatically satisfied).

Then necessarily also  $(B'_1)^0 = 0$ , for otherwise, in view of (9.11) we would have  $|u_1^c| \sim |(B'_1)^0| \neq 0$  when  $\tilde{\rho} = 0$ , which would contradict our assumption that  $u_1^0 = 0$ . Since  $((B'_1)^0, (\delta'_0)^0, (\delta'_{3,0})^0, (\delta'_4)^0) \in \Sigma$ , we thus see that  $((\delta'_{3,0})^0)^{\frac{4}{3}} + ((\delta'_4)^0)^2 = 1$ .

Therefore at the “base point”  $((B'_1)^0, (\delta'_0)^0, (\delta'_{3,0})^0, (\delta'_4)^0)$  the function  $\phi$  satisfies for  $\tilde{\rho} = 0$  the inequality

$$\sum_{j=2}^3 |\partial_{u_2}^j \phi(0, u_2^0, 0, \delta, s_2^0)| \neq 0,$$

and this inequality will persist for parameters sufficiently close to this base point.

Assume first that we even have  $\partial_{u_2}^2 \phi(0, u_2^0, 0, \delta, s_2^0) \neq 0$ . Then we can first apply the method of stationary phase to the  $u_2$  integration, and subsequently van der Corput's estimate in  $u_1$  (with  $N = 3$ ), which results in the estimate  $|I_0^\eta(\Lambda, \delta, s)| \leq C \Lambda^{-\frac{1}{2} - \frac{1}{3}}$ . This is again stronger than what we need.

There remains the case where  $\partial_{u_2}^2 \phi(0, u_2^0, 0, \delta, s_2^0) = 0$  and  $\partial_{u_2}^3 \phi(0, u_2^0, 0, \delta, s_2^0) \neq 0$ . In this case the phase function  $\Phi_2$  is a small smooth perturbation of a function  $\Phi_2^0$  of the form

$$\Phi_2^0(u_1, u_2) = c_3 u_1^3 + (u_2 - u_2^0)^3 b_3(u_2) + c_0,$$

where  $c_3 := B_3(s_2^0, \delta_1, 0) \neq 0$  and where  $b_3(u_2)$  is a smooth function such that  $b_3(u_2^0) \neq 0$ . This means that  $\Phi_2$  has a so-called  $D_4^+$ -type singularity in the sense of [3] and the distance between the associated Newton polyhedron and the origin is  $3/2$ . The estimate (9.10) therefore follows in this situation from the particular case of  $D_4^+$ -type singularities in Proposition 4.3.1 of [11].

Alternatively, one could also first treat the integration with respect to  $u_1$  by means of Lemma 6.3 in [21], with  $B = 3$ , and subsequently estimate the integration in  $u_2$  by means of van der Corput's lemma (we leave the details to the interested reader).

This concludes the proof of Lemma 9.1.

Q.E.D.

**Remark 9.2.** Notice that our phase  $\Phi$  in Lemma 9.1 is a small perturbation of a phase of the form  $c_1 x_1^3 + c_2 x_2^4$ , with  $c_1 \neq 0 \neq c_2$ , at least if we assume that  $|B_1(s, \delta_1)| \ll 1$  (this has been the interesting case the preceding proof). It is, however, not true that for arbitrary perturbations of such a phase function with a small perturbation parameter  $\delta > 0$  an estimate analogous to (9.4) of order  $O(c(\delta)\lambda^{-2/3})$  as  $\lambda \rightarrow \infty$  holds true. A counter example is given by the following function

$$\Phi(x, \delta) := x_1^3 + (x_2 - \delta)^4 + 4\delta(x_2 - \delta)^3 - 3\sqrt[3]{4\delta^2}x_1(x_2 - \delta)^2 + C(\delta),$$

where  $C(\delta)$  is chosen such that  $\Phi(0, \delta) \equiv 0$ . Note that  $\Phi(x, 0) = x_1^3 + x_2^4$ .

To see this, consider an oscillatory integral  $J(\lambda, \delta) := \int e^{i\lambda\Phi(x, \delta)} a(x) dx$  with phase function  $\Phi$ , whose amplitude is supported in a sufficiently small neighborhood of the origin and such that  $a(0) = 1$ .

When  $\delta > 0$  is sufficiently small, then  $\Phi$  has exactly two critical points, namely the degenerate critical point  $x_d := (0, \delta)$  and the non-degenerate critical point  $x_{nd} := (6\sqrt[3]{2}\delta^{4/3}, -6\delta)$ .

Let us consider the contribution of the degenerate critical point  $x_d$  to the oscillatory integral. The linear change of variables

$$z_1 = x_1 - \sqrt[3]{2\delta}(x_2 - \delta), \quad z_2 = x_1 + 2\sqrt[3]{2\delta}(x_2 - \delta)$$

transforms  $x_d$  into  $z_d = (0, 0)$  and the phase function into  $\tilde{\Phi}(z) + C(\delta)$ , where

$$\tilde{\Phi}(z) := z_1^2 z_2 + \left( \frac{z_2 - z_1}{3\sqrt[3]{2\delta}} \right)^4.$$

A look at the Newton polyhedron of  $\tilde{\Phi}$  reveals that the principal face of  $\mathcal{N}(\tilde{\Phi})$  is given by the compact edge  $[(0, 4), (2, 1)]$  which lies on the line given by  $\kappa_1 t_1 + \kappa_2 t_2 = 1$ , with associated weight  $\kappa = (\kappa_1, \kappa_2) := (3/8, 1/4)$ , and the principal part of  $\tilde{\Phi}$  is given by

$$\tilde{\Phi}_{\text{pr}}(z) = z_1^2 z_2 + \frac{z_2^4}{81\sqrt[3]{16\delta^4}}.$$



Moreover, the Newton distance is given by  $d = 8/5$ , whereas the non-trivial roots of  $\tilde{\Phi}_{\text{pr}}$  have multiplicity 1. Therefore, by Theorem 3.3 in [18] the coordinates  $(z_1, z_2)$  are adapted to  $\tilde{\Phi}$  in a sufficiently small neighborhood of the origin, so that the height  $h$  of  $\tilde{\Phi}$  in the sense of Varchenko is also given by  $h = d = 8/5$ . This implies that for every sufficiently small, fixed  $\delta > 0$  we have that

$$J(\lambda, \delta) = C(\delta)\lambda^{-\frac{5}{8}} + O\left(\lambda^{-\frac{7}{8}}\right) \quad \text{as } \lambda \rightarrow \infty,$$

with a non-trivial constant  $C(\delta)$ , because the contribution of the non-degenerate critical point  $x_{nd}$  is of order  $O(\lambda^{-1})$  (compare for instance [20]). This shows that an estimate of the type  $|J(\lambda, \delta)| \leq C(\delta)\lambda^{-2/3}$  can not hold in this example.

#### 10. THE CASE WHERE $m = 2$ , $B = 4$ AND $A = 0$

Again, we are assuming that  $\lambda\rho \gg 1$ , where  $\rho = \rho(\tilde{\delta})$  is given by (8.18). Observe that if  $A = 0$ , then according to (6.21) we have  $h^r + 1 = 4(n+3)/(n+4)$ , where  $n \geq 9$ , so that

$$(10.1) \quad \theta_c \leq \frac{13}{48}$$

Here we shall again perform a frequency decomposition near the Airy cone, by defining functions  $\nu_{\delta, Ai}^\lambda$  and  $\nu_{\delta, l}^\lambda$  as follows:

$$\begin{aligned} \widehat{\nu_{\delta, Ai}^\lambda}(\xi) &:= \chi_0(\rho^{-\frac{2}{3}}B_1(s, \delta_1))\widehat{\nu_\delta^\lambda}(\xi), \\ \widehat{\nu_{\delta, l}^\lambda}(\xi) &:= \chi_1((2^l\rho)^{-\frac{2}{3}}B_1(s, \delta_1))\widehat{\nu_\delta^\lambda}(\xi), \quad M_0 \leq 2^l \leq \frac{\rho^{-1}}{M_1}, \end{aligned}$$

so that

$$(10.2) \quad \nu_\delta^\lambda = \nu_{\delta, Ai}^\lambda + \sum_{\{l: M_0 \leq 2^l \leq \frac{\rho^{-1}}{M_1}\}} \nu_{\delta, l}^\lambda.$$

Denote by  $T_{\delta, Ai}^\lambda$  and  $T_{\delta, l}^\lambda$  the corresponding operators of convolution with the Fourier transforms of these functions.

**10.1. Estimation of  $T_{\delta, Ai}^\lambda$ .** Here we have  $|\rho^{-2/3}B_1(s, \delta_1)| \lesssim 1$ . In this case, we use the change of variables  $x =: \sigma_\rho u := (\rho^{1/3}u_1, \rho^{1/4}u_2)$  in the integral (8.15) defining  $\widehat{\nu_\delta}$ , and obtain

$$(10.3) \quad \widehat{\nu_\delta}(\xi) = \rho^{\frac{7}{12}} e^{-i\lambda s_3 B_0(s, \delta_1)} \int_{|\sigma_\rho u| < \varepsilon} e^{-i\lambda \rho s_3 \Phi_1(u, s, \delta)} a(\sigma_\rho u, \delta, s) du,$$

where the phase  $\Phi_1$  has the form

$$(10.4) \quad \begin{aligned} \Phi_1(u, s, \delta) &= u_1^3 B_3(s_2, \delta_1, \rho^{\frac{1}{3}}u_1) - u_1(\rho^{-\frac{2}{3}}B_1(s, \delta_1)) \\ &+ u_2^4 b(\sigma_\rho u, \delta_0^r, s_2) + \delta_4' u_2^2 \tilde{\alpha}_2(\rho^{\frac{1}{3}}u_1, \delta_0^r, s_2) + \delta_{3,0}' u_2 \tilde{\alpha}_1(\delta_0^r, s_2) + \delta_0' u_1 u_2 \alpha_{1,1}(\rho^{\frac{1}{3}}u_1, \delta_0^r, s_2), \end{aligned}$$

and where according to Remark 8.5 (in which we choose  $r := 1/\rho$ )  $\rho(\tilde{\delta}') = 1$ , so that

$$\delta'_0 + \delta'_{3,0} + \delta'_4 \sim 1$$

(recall that the coefficient  $\delta'_{3,0}$  does not appear in Case ND, where  $\alpha_{1,1} = 0$ ). We have also indicated that the amplitude  $a(\sigma_\rho u, \delta, s)$  is supported where  $|\sigma_\rho u| < \varepsilon$ , where we may assume that  $\varepsilon > 0$  is sufficiently small, since this will become important soon.

We shall proceed in a somewhat similar way as in Section 7 of [21], by choosing a cut-off function  $\chi_0 \in C_0^\infty(\mathbb{R}^2)$  such that  $\chi_0(u) = 1$  for  $|u| \leq R$ , where  $R$  will be chosen sufficiently large, and further decomposing

$$\widehat{\nu_\delta}(\xi) = \widehat{\nu_{\delta,0}}(\xi) + \widehat{\nu_{\delta,\infty}}(\xi),$$

where

$$\widehat{\nu_{\delta,0}}(\xi) := \rho^{\frac{7}{12}} e^{-i\lambda s_3 B_0(s, \delta_1)} \int e^{-i\lambda \rho s_3 \Phi_1(u, s, \delta)} a(\sigma_\rho u, \delta, s) \chi_0(u) du,$$

and

$$\widehat{\nu_{\delta,\infty}}(\xi) := \rho^{\frac{7}{12}} e^{-i\lambda s_3 B_0(s, \delta_1)} \int e^{-i\lambda \rho s_3 \Phi_1(u, s, \delta, s)} a(\sigma_\rho u, \delta, s) (1 - \chi_0(u)) du.$$

Accordingly, we decompose

$$\nu_{\delta, Ai}^\lambda = \nu_{\delta,0}^\lambda + \nu_{\delta,\infty}^\lambda,$$

where we have put

$$\begin{aligned} \widehat{\nu_{\delta,0}^\lambda}(\xi) &:= \chi_0(\rho^{-\frac{2}{3}} B_1(s, \delta_1)) \chi_1(s, s_3) \widehat{\nu_{\delta,0}}(\xi), \\ \widehat{\nu_{\delta,\infty}^\lambda}(\xi) &:= \chi_0(\rho^{-\frac{2}{3}} B_1(s, \delta_1)) \chi_1(s, s_3) \widehat{\nu_{\delta,\infty}}(\xi). \end{aligned}$$

Recall from (8.30) that  $\chi_1(s, s_3) = \chi_1(s_1, s_2, s_3)$  localizes to the region where  $s_j \sim 1, j = 1, 2, 3$ . The corresponding operators of convolution with  $\widehat{\nu_{\delta,0}^\lambda}$  and  $\widehat{\nu_{\delta,\infty}^\lambda}$  will be denoted by  $T_{\delta,0}^\lambda$  and  $T_{\delta,\infty}^\lambda$ , respectively.

Let us first consider the operators  $T_{\delta,\infty}^\lambda$ : By means of integrations by parts, we easily see that if  $R$  is chosen sufficiently large, then the phase will have no critical point, and thus for every  $N \in \mathbb{N}$  we have

$$(10.5) \quad \|\widehat{\nu_{\delta,\infty}^\lambda}\|_\infty \lesssim \rho^{\frac{7}{12}} (\lambda \rho)^{-N}.$$

Moreover, by Fourier inversion we find that

$$(10.6) \quad \nu_{\delta,\infty}^\lambda(x) = \lambda^3 \int_{\mathbb{R}^3} e^{i\lambda s_3(s_1 x_1 + s_2 x_2 + x_3)} \widehat{\nu_{\delta,\infty}^\lambda}(\xi) ds$$

(with  $\xi = \lambda s_3(s_1, s_2, 1)$ ). We then use the change of variables from  $s = (s_1, s_2)$  to  $(z, s_2)$ , where

$$z := \rho^{-\frac{2}{3}} B_1(s, \delta_1),$$

and find that (compare (8.12))

$$(10.7) \quad s_1 = s_2^{\frac{n-1}{n-2}} G_3(s_2, \delta) - \rho^{\frac{2}{3}} z,$$

and in particular

$$(10.8) \quad B_0(s, \delta, \sigma) = -\rho^{\frac{2}{3}} z s_2^{\frac{1}{n-2}} G_1(s_2, \delta) + s_2^{\frac{n}{n-2}} G_5(s_2, \delta).$$

And, if we plug the previous formula for  $\widehat{\nu_{\delta, \infty}^\lambda}$  into (10.6), we see that we may write  $\nu_{\delta, \infty}^\lambda(x)$  as an oscillatory

$$(10.9) \quad \nu_{\delta, \infty}^\lambda(x) = \rho^{\frac{7}{12} + \frac{2}{3}} \lambda^3 \int e^{-i\lambda s_3 \Phi_2(u, z, s_2, \delta)} \chi_0(z) (1 - \chi_0(u)) a(\sigma_\rho u, \rho^{\frac{2}{3}} z, s, \delta) \tilde{\chi}_1(s_2, s_3) du dz ds_2 ds_3$$

with respect to the variables  $u_1, u_2, z, s_2, s_3$ , where the complete phase is given by

$$(10.10) \quad \begin{aligned} \Phi_2(u, z, s_2, \delta) &:= s_2^{\frac{n}{n-2}} G_5(s_2, \delta) - x_1 s_2^{\frac{n-1}{n-2}} G_3(s_2, \delta) \\ &- s_2 x_2 - x_3 + \rho^{\frac{2}{3}} z \left( x_1 - s_2^{\frac{1}{n-2}} G_1(s_2, \delta) \right) + \rho \Phi_1(u, z, s_2, \delta), \end{aligned}$$

where according to (10.4), the phase  $\Phi_1$  is given in the new coordinates by

$$(10.11) \quad \begin{aligned} \Phi_1(u, z, s_2, \delta) &= u_1^3 B_3(s_2, \delta_1, \rho^{\frac{1}{3}} u_1) - u_1 z \\ &+ u_2^4 b(\sigma_\rho u, \delta_0^r, s_2) + \delta_4' u_2^2 \tilde{\alpha}_2(\rho^{\frac{1}{3}} u_1, \delta_0^r, s_2) + \delta_{3,0}' u_2 \tilde{\alpha}_1(\delta_0^r, s_2) + \delta_0' u_1 u_2 \alpha_{1,1}(\rho^{\frac{1}{3}} u_1, \delta_0^r, s_2). \end{aligned}$$

Recall also from (8.14) that  $|G_5| \sim 1 \sim |G_3|$  (where  $G_5 = G_1 G_3 - G_2$ ). The new amplitude  $a$  is again a smooth function of its arguments, and  $\tilde{\chi}_1(s_2, s_3)$  localizes to the region where  $|s_2| \sim 1 \sim |s_3|$ . Observe also that here  $|z| \lesssim 1$  and  $|\rho^{1/3} u_1| \leq \varepsilon$ ,  $|\rho^{1/4} u_2| \leq \varepsilon$ , so that the sum of the last two terms in  $\Phi_2$  can be viewed as a small error term of order  $O(\rho^{2/3} + \varepsilon)$ , provided  $|x| \lesssim 1$ .

Applying first again  $N$  integrations by parts with respect to the variables  $u_1, u_2$ , and then van der Corput's lemma to the integration in  $s_2$ , we thus find that

$$|\nu_{\delta, \infty}^\lambda(x)| \lesssim \rho^{\frac{7}{12} + \frac{2}{3}} \lambda^3 (\lambda \rho)^{-N} \lambda^{-\frac{1}{3}},$$

if  $|x| \lesssim 1$ .

However, if  $|x| \gg 1$ , then we may argue as in Section 7.2 of [21]: If  $|x_1| \ll |(x_2, x_3)|$ , then we easily see by means a further integration by parts with respect to the variables  $s_2$  or  $s_3$  that  $|\nu_{\delta, \infty}^\lambda(x)| \lesssim \rho^{\frac{7}{12} + \frac{2}{3}} \lambda^3 (\lambda \rho)^{-N} \lambda^{-1}$ , and if  $|x_1| \gtrsim |(x_2, x_3)|$ , then an integration by parts in  $z$  leads to  $|\nu_{\delta, \infty}^\lambda(x)| \lesssim \rho^{\frac{7}{12} + \frac{2}{3}} \lambda^3 (\lambda \rho)^{-N} (\lambda \rho^{2/3})^{-1}$ . Both these estimates are stronger than the previous one, and so altogether we have shown that

$$(10.12) \quad \|\nu_{\delta, \infty}^\lambda\|_\infty \lesssim \rho^{\frac{7}{12} + \frac{2}{3}} \lambda^{\frac{8}{3}} (\lambda \rho)^{-N}.$$

Interpolating between this estimate and (10.5), we obtain

$$\|T_{\delta, \infty}^\lambda\|_{p_c \rightarrow p'_c} \lesssim \rho^{\frac{7}{12}} (\lambda \rho)^{-N} \rho^{\frac{2\theta_c}{3}} \lambda^{\frac{8\theta_c}{3}},$$

which implies the desired estimate

$$\sum_{\lambda\rho\gg 1} \|T_{\delta,\infty}^\lambda\|_{p_c\rightarrow p'_c} \lesssim \rho^{\frac{7}{12}-2\theta_c} \leq 1,$$

since  $\theta_c \leq 13/48$ .

We next turn to the main terms  $\nu_{\delta,0}^\lambda$  and the corresponding operators  $T_{\delta,0}^\lambda$ . First we claim that

$$(10.13) \quad \|\widehat{\nu_{\delta,0}^\lambda}\|_\infty \lesssim \rho^{\frac{7}{12}}(\lambda\rho)^{-\frac{2}{3}}.$$

This is an immediate consequence of Lemma 9.1. Indeed our phase  $\Phi_1$  in (10.4) is of the form required in this lemma, with  $\rho$  in the lemma of size 1, but  $\lambda$  in the lemma replaced by  $\lambda\rho$  here, so that the oscillatory integral in the definition of  $\widehat{\nu_{\delta,0}^\lambda}$  can be estimated by  $C/(1^{1/12}(\lambda\rho)^{2/3})$ .

We finally want to estimate  $\|\nu_{\delta,0}^\lambda\|_\infty$ . In analogy with (10.9), we may write  $\nu_{\delta,0}^\lambda(x)$  as an oscillatory integral of the form

$$\nu_{\delta,0}^\lambda(x) = \rho^{\frac{7}{12}+\frac{2}{3}}\lambda^3 \int e^{-i\lambda s_3\Phi_2(u,z,s_2,\delta)} \chi_0(z)\chi_0(u)\tilde{a}(\sigma_\rho u, \rho^{\frac{2}{3}}z, s, \delta)\tilde{\chi}_1(s_2, s_3) dudzds_2ds_3,$$

with  $\Phi_2$  given by (10.10). We can reduce this to the following situation in which the amplitude is independent of  $z$ , i.e., where

$$(10.14) \quad \begin{aligned} \nu_{\delta,0}^\lambda(x) &= \rho^{\frac{7}{12}+\frac{2}{3}}\lambda^3 \int e^{-i\lambda s_3\Phi_2(u,z,s_2,\delta)} \\ &\quad \times \chi_0(z)\chi_0(u)a(\sigma_\rho u, s, \rho^{\frac{1}{3}}, \delta)\tilde{\chi}_1(s_2, s_3) dudzds_2ds_3. \end{aligned}$$

In fact, we may develop the amplitude  $\tilde{a}$  into a convergent series of smooth functions, each of which is a tensor product of a smooth function of the variable  $z$  with a smooth function depending on the remaining variables only. Thus, by considering each of the corresponding terms separately, we can reduce to the situation (10.14) (the function  $\chi_0(z)$  will of course have to be different from the previous one).

We claim that

$$(10.15) \quad \|\nu_{\delta,0}^\lambda\|_\infty \lesssim \rho^{\frac{7}{12}}\lambda^2(\lambda\rho)^{-\frac{1}{4}}.$$

Indeed, if  $|x| \gg 1$ , arguing in a similar way as for  $\nu_{\delta,\infty}^\lambda(x)$ , we see that  $|\nu_{\delta,0}^\lambda(x)| \lesssim \rho^{\frac{7}{12}+\frac{2}{3}}\lambda^3(\lambda\rho^{2/3})^{-N}$  for every  $N \in \mathbb{N}$ , which is stronger than what is needed for (10.15).

From now on we shall therefore assume that  $|x| \lesssim 1$ . For such  $x$  fixed, we can argue in a similar way as in in Section 7.2 of [21] (compare also with Subsection 12.2): We decompose

$$(10.16) \quad \nu_{\delta,0}^\lambda = \nu_{0,I}^\lambda + \nu_{0,II}^\lambda,$$

where  $\nu_{0,I}^\lambda$  and  $\nu_{0,II}^\lambda$  denote the contributions to the integral (10.14) by the region  $L_I$  given by

$$|x_1 - s_2^{\frac{1}{n-2}} G_1(s_2, \delta)| \gg \rho^{\frac{1}{3}},$$

and the region  $L_{II}$  where

$$|x_1 - s_2^{\frac{1}{n-2}} G_1(s_2, \delta)| \lesssim \rho^{\frac{1}{3}},$$

respectively. Recall from (10.10) and (10.11) that

$$\begin{aligned} \Phi_2(u, z, s_2, \delta) &= s_2^{\frac{n}{n-2}} G_5(s_2, \delta) - x_1 s_2^{\frac{n-1}{n-2}} G_3(s_2, \delta) - s_2 x_2 - x_3 \\ (10.17) \quad &+ z \rho \left( \rho^{-\frac{1}{3}} (x_1 - s_2^{\frac{1}{n-2}} G_1(s_2, \delta)) - u_1 \right) + \rho u_1^3 B_3(s_2, \delta_1, \rho^{\frac{1}{3}} u_1) \\ &+ \rho \left( u_2^4 b(\sigma_\rho u, \delta_0^\tau, s_2) + \delta_4' u_2^2 \tilde{\alpha}_2(\rho^{\frac{1}{3}} u_1, \delta_0^\tau, s_2) + \delta_{3,0}' u_2 \tilde{\alpha}_1(\delta_0^\tau, s_2) + \delta_0' u_1 u_2 \alpha_{1,1}(\rho^{\frac{1}{3}} u_1, \delta_0^\tau, s_2) \right). \end{aligned}$$

Let us change variables from  $s_2$  first to  $v := x_1 - s_2^{\frac{1}{n-2}} G_1(s_2, \delta)$ , and then to  $w := \rho^{-\frac{1}{3}} v = \rho^{-\frac{1}{3}} (x_1 - s_2^{\frac{1}{n-2}} G_1(s_2, \delta))$ . In these new coordinates,  $\Phi_2$  can be written as

$$\Phi_2 = z \rho (w - u_1) + \Phi_3,$$

with  $\Phi_3$  of the form

$$\begin{aligned} \Phi_3(u, w, x, \delta) &= \Psi_3(\rho^{\frac{1}{3}} w, x, \delta) + \rho u_1^3 B_3(\rho^{\frac{1}{3}} w, \rho^{\frac{1}{3}} u_1, x, \delta) \\ (10.18) \quad &+ \rho \left( u_2^4 b(\sigma_\rho u, \rho^{\frac{1}{3}} w, x, \delta_0^\tau) + \delta_4' u_2^2 \tilde{\alpha}_2(\rho^{\frac{1}{3}} u_1, \rho^{\frac{1}{3}} w, x, \delta_0^\tau) \right. \\ &\left. + \delta_{3,0}' u_2 \tilde{\alpha}_1(\rho^{\frac{1}{3}} w, x, \delta_0^\tau) + \delta_0' u_1 u_2 \alpha_{1,1}(\rho^{\frac{1}{3}} u_1, \rho^{\frac{1}{3}} w, x, \delta_0^\tau) \right), \end{aligned}$$

where  $\Psi_3$  is a smooth, real-valued function. With a slight abuse of notation, we have here used the same symbols  $B_3, b, \dots, \alpha_{1,1}$  as before, since these functions will have the same basic properties here as the corresponding ones in (10.17). A similar remark will apply to the amplitudes, which we shall always denote by the letter  $a$ , even though they may change form line to line. Moreover, we may write

$$\begin{aligned} \nu_{0,I}^\lambda(x) &= (\rho^{\frac{7}{12} + \frac{2}{3}} \lambda^3) \rho^{\frac{1}{3}} \int e^{-i\lambda s_3 \Phi_3(u, w, x, \delta)} \widehat{\chi_0}(\lambda \rho s_3 (w - u_1)) (1 - \chi_0(w)) \chi_0(u) \\ (10.19) \quad &\times a(\sigma_\rho u, \rho^{\frac{1}{3}} w, s_1, \rho^{\frac{1}{3}}, x, \delta) \tilde{\chi}_1(s_3) dw du ds_3. \end{aligned}$$

$$\begin{aligned} \nu_{0,II}^\lambda(x) &= (\rho^{\frac{7}{12} + \frac{2}{3}} \lambda^3) \rho^{\frac{1}{3}} \int e^{-i\lambda s_3 \Phi_3(u, w, x, \delta)} \widehat{\chi_0}(\lambda \rho s_3 (w - u_1)) \chi_0(w) \chi_0(u) \\ (10.20) \quad &\times a(\sigma_\rho u, \rho^{\frac{1}{3}} w, s_1, \rho^{\frac{1}{3}}, x, \delta) \tilde{\chi}_1(s_3) dw du ds_3. \end{aligned}$$

Here,  $\chi_0(w)$  will again denote a smooth function with compact support which is identically 1 on a sufficiently large neighborhood of the origin.

Observe that in (10.19) we have  $|w| \gg 1 \gtrsim |u_1|$ , so that  $|\widehat{\chi_0}(\lambda \rho s_3 (w - u_1))| \leq C_N (\lambda \rho |w|)^{-(N+1)}$  for every  $N \in \mathbb{N}$ , and we immediately obtain the estimate

$$(10.21) \quad \|\nu_{0,I}^\lambda(x)\|_\infty \leq C_N (\rho^{\frac{7}{12} + \frac{2}{3}} \lambda^3) \rho^{\frac{1}{3}} (\lambda \rho)^{-(N+1)} = \rho^{\frac{7}{12}} \lambda^2 (\lambda \rho)^{-N},$$

which is even stronger than (10.15).

In order to estimate the second term, we perform yet another change of variables from  $u_1$  to  $y_1$  so that  $u_1 = w - (\lambda\rho)^{-1}y_1$ , i.e.,  $y_1 = \lambda\rho(w - u_1)$ . This leads to the following expression for  $\nu_{0,II}^\lambda(x)$  :

$$(10.22) \quad \begin{aligned} \nu_{0,II}^\lambda(x) &= (\rho^{\frac{7}{12}+\frac{2}{3}}\lambda^3) \rho^{\frac{1}{3}} (\lambda\rho)^{-1} \int e^{-i\lambda s_3 \Phi_4(y_1, u_2, w, x, \delta)} \widehat{\chi_0}(s_3 y_1) \chi_0(w) \chi_0(w - (\lambda\rho)^{-1}y_1) \\ &\quad \times \chi_0(u_2) a_4((\lambda\rho^{\frac{2}{3}})^{-1}y_1, \rho^{\frac{1}{4}}u_2, w, s_1, \rho^{\frac{1}{3}}, x, \delta) \tilde{\chi}_1(s_3) dy_1 du_2 dw ds_3, \end{aligned}$$

with phase  $\Phi_4$  of the form

$$(10.23) \quad \begin{aligned} \Phi_4(y_1, u_2, w, x, \delta) &= \Psi_3(\rho^{\frac{1}{3}}w, x, \delta) + \rho(w - (\lambda\rho)^{-1}y_1)^3 \tilde{B}_3(\rho^{\frac{1}{3}}w, (\lambda\rho^{\frac{2}{3}})^{-1}y_1, x, \delta) \\ &\quad + \rho \left( u_2^4 b(\rho^{\frac{1}{3}}w, (\lambda\rho^{\frac{2}{3}})^{-1}y_1, \rho^{\frac{1}{4}}u_2, x, \delta_0^r) + \delta'_4 u_2^2 \tilde{\alpha}_2(\rho^{\frac{1}{3}}w, (\lambda\rho^{\frac{2}{3}})^{-1}y_1, \rho^{\frac{1}{4}}u_2, x, \delta_0^r) \right. \\ &\quad \left. + \delta'_{3,0} u_2 \tilde{\alpha}_1(\rho^{\frac{1}{3}}w, x, \delta_0^r) + \delta'_0 u_2 (w - (\lambda\rho)^{-1}y_1) \alpha_{1,1}(\rho^{\frac{1}{3}}w, (\lambda\rho^{\frac{2}{3}})^{-1}y_1, x, \delta_0^r) \right). \end{aligned}$$

Observe that in this integral,  $|u_2| + |w| \lesssim 1$  and  $|y_1| \lesssim \lambda\rho$ . Moreover, the factor  $\widehat{\chi_0}(s_3 y_1)$  guarantees the absolute convergence of this integral with respect to the variable  $y_1$ . We can thus first apply van der Corput's estimate of order  $M = 4$  for the integration in  $u_2$ , which leads to an additional factor of order  $(\lambda\rho)^{-1/4}$ , and then perform the remaining integrations in  $w, y_1$  and  $s_3$ . Altogether, this leads to the estimate

$$(10.24) \quad \|\nu_{0,I}^\lambda(x)\|_\infty \leq C(\rho^{\frac{7}{12}+\frac{2}{3}}\lambda^3) \rho^{\frac{1}{3}} (\lambda\rho)^{-1} (\lambda\rho)^{-\frac{1}{4}} = \rho^{\frac{7}{12}} \lambda^2 (\lambda\rho)^{-\frac{1}{4}}.$$

In combination with (10.21), this proves (10.26).

Finally, interpolating between the estimates (10.13) and (10.15), we obtain

$$\|T_{\delta,0}^\lambda\|_{p_c \rightarrow p'_c} \lesssim \rho^{\frac{7}{12}-2\theta_c} (\lambda\rho)^{\frac{29}{12}\theta_c - \frac{2}{3}}.$$

But, since  $\theta_c \leq 13/48$ , we have  $\frac{29}{12}\theta_c - \frac{2}{3} < 0$  and  $7/12 - 2\theta_c > 0$ , which implies the desired estimate

$$\sum_{\lambda\rho \gg 1} \|T_{\delta,0}^\lambda\|_{p_c \rightarrow p'_c} \lesssim \rho^{\frac{7}{12}-2\theta_c} \leq 1.$$

Altogether, we have thus proved that

$$(10.25) \quad \sum_{\lambda\rho \gg 1} \|T_{\delta,Ai}^\lambda\|_{p_c \rightarrow p'_c} \lesssim 1.$$

10.2. **Estimation of  $T_{\delta,l}^\lambda$ .** Here we have  $|(2^l \rho)^{-2/3} B_1(s, \delta_1)| \sim 1$ . Recall also that  $2^l \rho \leq 1/M_1 \ll 1$ . In this case, we use the change of variables  $x =: \sigma_{2^l \rho} u := ((2^l \rho)^{1/3} u_1, (2^l \rho)^{1/4} u_2)$  in the integral (8.15) defining  $\widehat{\nu}_\delta$ , and obtain

$$(10.26) \quad \widehat{\nu}_\delta(\xi) = (2^l \rho)^{\frac{7}{12}} e^{-i\lambda s_3 B_0(s, \delta_1)} \int_{|\sigma_{2^l \rho} u| < \varepsilon} e^{-i\lambda 2^l \rho s_3 \Phi_1(u, s, \delta)} a(\sigma_{2^l \rho} u, \delta, s) du,$$

where now the phase  $\Phi_1 = \Phi_{1,l}$  has the form

$$(10.27) \quad \begin{aligned} \Phi_1(u, s, \delta) = & u_1^3 B_3(s_2, \delta_1, (2^l \rho)^{\frac{1}{3}} u_1) - u_1((2^l \rho)^{-\frac{2}{3}} B_1(s, \delta_1)) + u_2^4 b(\sigma_{2^l \rho} u, \delta_0^r, s_2) \\ & + \delta'_4 u_2^2 \tilde{\alpha}_2((2^l \rho)^{\frac{1}{3}} u_1, \delta_0^r, s_2) + \delta'_{3,0} u_2 \tilde{\alpha}_1(\delta_0^r, s_2) + \delta'_0 u_1 u_2 \alpha_{1,1}((2^l \rho)^{\frac{1}{3}} u_1, \delta_0^r, s_2), \end{aligned}$$

and where according to Remark 8.5 (in which we choose  $r := 1/(2^l \rho)$ )  $\rho(\tilde{\delta}') = 2^{-l}$ , so that

$$(\delta'_4)^2 + (\delta'_{3,0})^{\frac{4}{3}} + (\delta'_0)^{\frac{12}{5}} = 2^{-l} \leq \frac{1}{M_0} \ll 1.$$

(recall that the coefficient  $\delta'_{3,0}$  does not appear in Case ND, where  $\alpha_{1,1} = 0$ ). We have also indicated that the amplitude  $a(\sigma_{2^l \rho} u, \delta, s)$  is supported where  $|\sigma_{2^l \rho} u| < \varepsilon$ , and that we may assume that  $\varepsilon > 0$  is sufficiently small.

Observe that the second row in (10.27) is a small perturbation of the leading term, given by the first row.

Again, we choose a cut-off function  $\chi_0 \in C_0^\infty(\mathbb{R}^2)$  such that  $\chi_0(u) = 1$  for  $|u| \leq R$ , where  $R$  will be chosen sufficiently large, and further decompose

$$\widehat{\nu}_\delta(\xi) = \widehat{\nu}_{\delta,0}(\xi) + \widehat{\nu}_{\delta,\infty}(\xi),$$

where now

$$\widehat{\nu}_{\delta,0}(\xi) := (2^l \rho)^{\frac{7}{12}} e^{-i\lambda s_3 B_0(s, \delta_1)} \int_{|\sigma_{2^l \rho} u| < \varepsilon} e^{-i\lambda 2^l \rho s_3 \Phi_1(u, s, \delta)} a(\sigma_{2^l \rho} u, \delta, s) \chi_0(u) du,$$

and

$$\widehat{\nu}_{\delta,\infty}(\xi) := (2^l \rho)^{\frac{7}{12}} e^{-i\lambda s_3 B_0(s, \delta_1)} \int_{|\sigma_{2^l \rho} u| < \varepsilon} e^{-i\lambda 2^l \rho s_3 \Phi_1(u, s, \delta)} a(\sigma_{2^l \rho} u, \delta, s) (1 - \chi_0(u)) du,$$

Accordingly, we decompose

$$\nu_{\delta,l}^\lambda = \nu_{l,0}^\lambda + \nu_{l,\infty}^\lambda,$$

where we have put

$$\begin{aligned} \widehat{\nu_{l,0}^\lambda}(\xi) &:= \chi_1((2^l \rho)^{-\frac{2}{3}} B_1(s, \delta_1)) \chi_1(s, s_3) \widehat{\nu}_{\delta,0}(\xi), \\ \widehat{\nu_{l,\infty}^\lambda}(\xi) &:= \chi_1((2^l \rho)^{-\frac{2}{3}} B_1(s, \delta_1)) \chi_1(s, s_3) \widehat{\nu}_{\delta,\infty}(\xi). \end{aligned}$$

The corresponding operators of convolution with  $\widehat{\nu_{l,0}^\lambda}$  and  $\widehat{\nu_{l,\infty}^\lambda}$  will be denoted by  $T_{l,0}^\lambda$  and  $T_{l,\infty}^\lambda$ , respectively.

Let us again first consider the operators  $T_{l,\infty}^\lambda$ : By means of integrations by parts, we easily see that if  $R$  is chosen sufficiently large, then the phase will have no critical point, and thus for every  $N \in \mathbb{N}$  we have

$$(10.28) \quad \|\widehat{\nu_{l,\infty}^\lambda}\|_\infty \lesssim (2^l \rho)^{\frac{7}{12}} (\lambda 2^l \rho)^{-N}.$$

Moreover, Fourier inversion again leads to (10.6), and performing the change of variables from  $s = (s_1, s_2)$  to  $(z, s_2)$ , where here

$$z := (2^l \rho)^{-\frac{2}{3}} B_1(s, \delta_1),$$

we find that

$$(10.29) \quad s_1 = s_2^{\frac{n-1}{n-2}} G_3(s_2, \delta) - (2^l \rho)^{\frac{2}{3}} z,$$

and in particular

$$(10.30) \quad B_0(s, \delta, \sigma) = -(2^l \rho)^{\frac{2}{3}} z s_2^{\frac{1}{n-2}} G_1(s_2, \delta) + s_2^{\frac{n}{n-2}} G_5(s_2, \delta).$$

In a similar way as before, this leads to

$$(10.31) \quad \begin{aligned} \nu_{l,\infty}^\lambda(x) &= (2^l \rho)^{\frac{7}{12} + \frac{2}{3}} \lambda^3 \\ &\times \int e^{-i\lambda s_3 \Phi_2(u, z, s_2, \delta)} \chi_1(z) (1 - \chi_0(u)) a(\sigma_{2^l \rho} u, (2^l \rho)^{\frac{2}{3}} z, s, \delta) \tilde{\chi}_1(s_2, s_3) du dz ds_2 ds_3 \end{aligned}$$

with respect to the variables  $u_1, u_2, z, s_2, s_3$ , where the complete phase is now given by

$$(10.32) \quad \begin{aligned} \Phi_2(u, z, s_2, \delta) &:= s_2^{\frac{n}{n-2}} G_5(s_2, \delta) - x_1 s_2^{\frac{n-1}{n-2}} G_3(s_2, \delta) \\ &- s_2 x_2 - x_3 + (2^l \rho)^{2/3} z (x_1 - s_2^{\frac{1}{n-2}} G_1(s_2, \delta) + 2^l \rho \Phi_1(u, z, s_2, \delta)). \end{aligned}$$

According to (10.27), the phase  $\Phi_1$  is given in the new coordinates by

$$(10.33) \quad \begin{aligned} \Phi_1(u, z, s_2, \delta) &= u_1^3 B_3(s_2, \delta_1, (2^l \rho)^{\frac{1}{3}} u_1) - u_1 z + u_2^4 b(\sigma_{2^l \rho} u, \delta_0^\tau, s_2) \\ &+ \delta_4' u_2^2 \tilde{\alpha}_2((2^l \rho)^{\frac{1}{3}} u_1, \delta_0^\tau, s_2) + \delta_{3,0}' u_2 \tilde{\alpha}_1(\delta_0^\tau, s_2) + \delta_0' u_1 u_2 \alpha_{1,1}((2^l \rho)^{\frac{1}{3}} u_1, \delta_0^\tau, s_2). \end{aligned}$$

Arguing as in the preceding subsection, by applying first  $N$  integrations by parts with respect to the variables  $u_1, u_2$ , and then van der Corput's lemma (of order  $M = 3$ ) to the integration in  $s_2$ , we thus find that

$$|\nu_{l,\infty}^\lambda(x)| \lesssim (2^l \rho)^{\frac{7}{12} + \frac{2}{3}} \lambda^3 (\lambda 2^l \rho)^{-N} \lambda^{-\frac{1}{3}},$$

first if  $|x| \lesssim 1$ , and then also for  $|x| \gg 1$ , by the same kind of arguments. We thus obtain

$$(10.34) \quad \|\nu_{l,\infty}^\lambda\|_\infty \lesssim (2^l \rho)^{\frac{7}{12} + \frac{2}{3}} \lambda^{\frac{8}{3}} (\lambda 2^l \rho)^{-N}.$$

Interpolating between this estimate and (10.28), we find here that

$$\|T_{l,\infty}^\lambda\|_{p_c \rightarrow p_c'} \lesssim (2^l \rho)^{\frac{7}{12}} (\lambda 2^l \rho)^{-N} (2^l \rho)^{\frac{2\theta_c}{3}} \lambda^{\frac{8\theta_c}{3}},$$



which implies that

$$\sum_{\lambda\rho \gg 1} \|T_{l,\infty}^\lambda\|_{p_c \rightarrow p'_c} \lesssim 2^{(\frac{7}{12} + \frac{2\theta_c}{3} - N)l} \rho^{\frac{7}{12} - 2\theta_c} \leq 1,$$

hence the desired estimate

$$\sum_{\{l: M_0 \leq 2^l \leq \frac{\rho^{-1}}{M_1}\}} \sum_{\lambda\rho \gg 1} \|T_{l,\infty}^\lambda\|_{p_c \rightarrow p'_c} \lesssim \rho^{\frac{7}{12} - 2\theta_c} \leq 1,$$

since  $\theta_c \leq 13/48$ .

We next turn to the main terms  $\nu_{l,0}^\lambda$  and the corresponding operators  $T_{l,0}^\lambda$ . First we show that

$$(10.35) \quad \|\widehat{\nu_{l,0}^\lambda}\|_\infty \lesssim (2^l \rho)^{\frac{7}{12}} (\lambda 2^l \rho)^{-\frac{3}{4}}.$$

Indeed, (10.27) in combination with (8.3) shows that the phase  $\Phi_1$  is a small perturbation of the phase

$$\Phi_{1,0}(u, s) := u_1^3 B_3(s_2, 0, 0) - c_1 u_1 + u_2^4 b_4(0),$$

where  $c_1$  corresponds to a fixed value of  $(2^l \rho)^{-2/3} B_1(s, \delta_1)$ , so that  $|c_1| \sim 1$ . Now, this phase will either have a critical point with respect to the variable  $u_1$  (recall that  $u \in \text{supp } \chi_0$ ), and then we can apply the method of stationary phase to the  $u_1$ -integration, or, otherwise we may use integrations by parts in  $u_1$  (for instance, if  $c_1$  and  $B_3$  have opposite signs). Applying subsequently van der Corput's estimate (of order  $M = 4$ ) to the  $u_1$ -integration, we immediately get (10.35).

We finally want to estimate  $\|\nu_{l,0}^\lambda\|_\infty$ . In analogy with (10.31), we may write  $\nu_{l,0}^\lambda(x)$  as an oscillatory integral of the form

$$\begin{aligned} \nu_{l,0}^\lambda(x) &= (2^l \rho)^{\frac{7}{12} + \frac{2}{3}} \lambda^3 \\ &\times \int e^{-i\lambda s_3 \Phi_2(u, z, s_2, \delta)} \chi_1(z) \chi_0(u) a(\sigma_{2^l \rho} u, (2^l \rho)^{\frac{2}{3}} z, s, \delta) \tilde{\chi}_1(s_2, s_3) du dz ds_2 ds_3 \end{aligned}$$

with  $\Phi_2$  given by (10.32). We can then basically argue as we did for  $\nu_{\delta, Ai}^\lambda$  in the previous subsection, only with  $\rho$  replaced by  $2^l \rho$ , and arrive correspondingly at the following analogue of estimate (10.15):

$$(10.36) \quad \|\nu_{l,0}^\lambda\|_\infty \lesssim (2^l \rho)^{\frac{7}{12}} \lambda^2 (\lambda 2^l \rho)^{-\frac{1}{4}}.$$

Finally, interpolating between the estimates (10.35) and (10.36), we obtain

$$(10.37) \quad \|T_{l,0}^\lambda\|_{p_c \rightarrow p'_c} \lesssim (2^l \rho)^{\frac{7}{12}} \lambda^{2\theta_c} (\lambda 2^l \rho)^{\frac{\theta_c}{2} - \frac{3}{4}} = 2^{\frac{l}{6}(3\theta_c - 1)} (\lambda \rho)^{\frac{10\theta_c - 3}{4}} \rho^{\frac{7}{12} - 2\theta_c}.$$

But, since  $\theta_c \leq 13/48$ , we have  $10\theta_c - 3 < 0$  and  $3\theta_c - 1 < 0$ , which implies the desired estimate

$$\sum_{\{l: M_0 \leq 2^l \leq \frac{\rho^{-1}}{M_1}\}} \sum_{\lambda\rho \gg 1} \|T_{l,0}^\lambda\|_{p_c \rightarrow p'_c} \lesssim 1.$$

Altogether, we have thus proved that

$$(10.38) \quad \sum_{\{l: M_0 \leq 2^l \leq \frac{\rho-1}{M_1}\}} \sum_{\lambda \rho \gg 1} \|T_{\delta, l}^\lambda\|_{p_c \rightarrow p'_c} \lesssim 1.$$

This completes the proof of Proposition 8.8 also for the case where  $A = 0$ .

#### 11. THE CASE WHERE $m = 2$ , $B = 3$ AND $A = 0$ : WHAT STILL NEEDS TO BE DONE

What remains open in (6.20), hence also in the proof of Proposition 2.1, is the case where  $B = 3$  and  $A = 0$ , in which  $\theta_c = 1/3$  and  $p_c = 6/5$ . Notice also that in this case (8.18) means that

$$\rho := \begin{cases} \delta_{3,0}^{\frac{3}{2}} & \text{in Case ND,} \\ \delta_0^3 + \delta_{3,0}^{\frac{3}{2}} & \text{in Case D,} \end{cases}$$

where in Case ND  $\delta_{3,0} \geq \delta_0$ . This shows that  $\rho \geq \delta_0^3$  in both cases.

Let us first observe that estimate (6.5) shows that we “trivially” have

$$\sum_{\lambda \gtrsim \delta_0^{-3}} \|T_\delta^\lambda\|_{p_c \rightarrow p'_c} \lesssim 1,$$

since  $B = 3$  and  $\theta_c = 1/3$ . In the sequel, we may and shall therefore always assume that  $\lambda \ll \delta_0^{-3}$ .

According to our discussion in Subsection 8.2, if  $\lambda \rho \lesssim 1$  (where  $\rho = \rho(\tilde{\delta})$ ), the endpoint  $p = p_c$  is still left open.

On the other hand, if  $\lambda \rho \gg 1$ , we can basically follow our approach from the previous section, with only minor modifications. Let us describe some more details.

Again, we perform the “Airy-cone decomposition” (10.2). In order to estimate  $T_{\delta, Ai}^\lambda$  we may here use the scaling  $x =: \sigma_\rho u := (\rho^{1/3} u_1, \rho^{1/3} u_2)$ , which leads to

$$(11.1) \quad \widehat{\nu}_\delta(\xi) = \rho^{\frac{2}{3}} e^{-i\lambda s_3 B_0(s, \delta_1)} \int_{|\sigma_\rho u| < \varepsilon} e^{-i\lambda \rho s_3 \Phi_1(u, s, \delta)} a(\sigma_\rho u, \delta, s) du,$$

where the phase  $\Phi_1$  now has the form

$$(11.2) \quad \begin{aligned} \Phi_1(u, s, \delta) &= u_1^3 B_3(s_2, \delta_1, \rho^{\frac{1}{3}} u_1) - u_1(\rho^{-\frac{2}{3}} B_1(s, \delta_1)) \\ &\quad + u_2^3 b(\sigma_\rho u, \delta_0^r, s_2) + \delta'_{3,0} u_2 \tilde{\alpha}_1(\delta_0^r, s_2) + \delta'_0 u_1 u_2 \alpha_{1,1}(\rho^{\frac{1}{3}} u_1, \delta_0^r, s_2), \end{aligned}$$

in place of (10.3) and (10.4), and where

$$\delta'_0 + \delta'_{3,0} \sim 1.$$

Recall that the coefficient  $\delta'_0$  does not appear in Case ND, in which  $\alpha_{1,1} = 0$ . If we again decompose

$$\nu_{\delta, Ai}^\lambda = \nu_{\delta, 0}^\lambda + \nu_{\delta, \infty}^\lambda,$$

then one finds here that, in place of (10.5) and (10.12), we obtain  $\|\widehat{\nu_{\delta,\infty}^\lambda}\|_\infty \lesssim \rho^{2/3}(\lambda\rho)^{-N}$  and  $\|\nu_{\delta,\infty}^\lambda\|_\infty \lesssim \rho^{4/3}\lambda^{8/3}(\lambda\rho)^{-N}$ , for every  $N \in \mathbb{N}$ . By interpolation, this leads to

$$\|T_{\delta,\infty}^\lambda\|_{p_c \rightarrow p'_c} \lesssim (\lambda\rho)^{\frac{8}{9}-N},$$

which implies the desired estimate

$$(11.3) \quad \sum_{\lambda\rho \gg 1} \|T_{\delta,\infty}^\lambda\|_{p_c \rightarrow p'_c} \lesssim 1.$$

As for the main term  $\nu_{\delta,0}^\lambda$ , a similar type of discussion that led to (10.13) here yields the following estimate:

$$(11.4) \quad \|\widehat{\nu_{\delta,0}^\lambda}\|_\infty \lesssim \rho^{\frac{2}{3}}(\lambda\rho)^{-\frac{1}{2}-\frac{1}{3}} = \rho^{\frac{2}{3}}(\lambda\rho)^{-\frac{5}{6}}.$$

Indeed, recall that we are assuming that  $\lambda\rho \gg 1$  and  $\lambda \ll \delta_0^{-3}$ , so that  $\rho \gg \delta_0^3$ . The estimate (11.4) therefore follows from the following analog of Lemma 9.1 for the case where  $B = 3$ :

**Lemma 11.1.** *Denote by  $J(\lambda, \delta, s)$  the oscillatory*

$$J(\lambda, \delta, s) = \chi_1(s_1, s_2) \iint e^{-i\lambda\Phi(x,\delta,s)} a(x, \delta, s) dx_1 dx_2,$$

*with phase*

$$\Phi(x, \delta, s) = x_1^3 B_3(s_2, \delta_1, x_1) - x_1 B_1(s, \delta_1) + \phi^\sharp(x_1, x_2, \delta, s_2),$$

*where*

$$\phi^\sharp(x, \delta, s_2) := x_2^3 b(x, \delta_0^r, s_2) + \delta_{3,0} x_2 \tilde{\alpha}_1(x_1, \delta_0^r, s_2) + \delta_0 x_1 x_2 \alpha_{1,1}(x_1, \delta_0^r, s_2),$$

*and where  $\chi_1(s_1, s_2)$  localizes to the region where  $s_j \sim 1, j = 1, 2$ . Let us also put*

$$\tilde{\rho} := \rho + |B_1(s, \delta_1)|^{\frac{3}{2}},$$

*and assume that  $\tilde{\rho} \geq M\delta_0^3$ . Then the following estimate*

$$(11.5) \quad |J(\lambda, \delta, s)| \leq \frac{C}{\tilde{\rho}^{\frac{1}{6}} \lambda^{\frac{5}{6}}}$$

*holds true, provided the amplitude  $a$  is supported in a sufficiently small neighborhood of the origin and  $M \geq 1$  is sufficiently large. The constant  $C$  in this estimate is independent of  $\delta$  and  $s$ .*

**Remark 11.2.** *Without the assumption  $\tilde{\rho} \gg \delta_0^3$ , estimate (11.5) may fail. Indeed, in the worst case, it may happen that, after re-scaling, all of the quantities  $B'_1(s, \delta_1), \delta'_0$  and  $\delta'_{3,0}$  are of size 1, and a degenerate critical point of order 4 arises for the integration in  $u_2$ , after we have applied the method of stationary phase in  $u_1$ . In this case, we will only obtain an estimate  $|I(\Lambda, \delta, s)| \leq C\Lambda^{-1/2-1/4}$ , and this estimate will be sharp.*

*Proof.* The proof is analogous to the proof of Lemma 9.1. In the more difficult case where  $\Lambda := \tilde{\rho}\lambda \gtrsim 1$ , after applying the change of coordinates  $x = \tilde{\rho}^{1/3}u$ , we see that it suffices to prove an estimate of the form  $|I(\Lambda, \delta, s)| \leq C\Lambda^{-5/6}$ , where the oscillatory integral  $I(\Lambda, \delta, s)$  is as in (9.5), only with  $\phi$  in the phase  $\Phi_1$  replaced by

$$\phi(u, \tilde{\rho}, \delta, s_2) := u_2^3 b(\tilde{\rho}^{1/3}u, \delta_0^r, s_2) + \delta'_{3,0} u_2 \tilde{\alpha}_1(\tilde{\rho}^{1/3}u_1, \delta_0^r, s_2) + \delta'_0 u_1 u_2 \alpha_{1,1}(\tilde{\rho}^{1/3}u_1, \delta_0^r, s_2)$$

(compare (9.6)) Here,

$$B'_1(s, \delta_1) := \frac{B_1(s, \delta_1)}{\tilde{\rho}^{2/3}}, \quad \delta'_0 := \frac{\delta_0}{\tilde{\rho}^{1/3}}, \quad \delta'_{3,0} := \frac{\delta_{3,0}}{\tilde{\rho}^{2/3}}.$$

Notice that our assumption implies that  $\delta'_0 \ll 1$ , and then

$$(\delta'_{3,0})^{3/2} + |B'_1(s, \delta_1)|^{3/2} \sim 1.$$

This shows that the phase  $\Phi_1$  will have at worst an Airy type singularity in one of the variables  $u_1$  or  $u_2$ . Applying thus first the method of stationary phase to the integration in one of these variables, and subsequently van der Corput's estimates of order 3 to the integration in the second variable, we arrive at the desired estimate for  $I(\Lambda, \delta, s)$ . Q.E.D.

Next, in order to estimate  $\nu_{\delta,0}^\lambda(x)$ , recall that

$$(11.6) \quad \nu_{\delta,0}^\lambda(x) = \rho^{2/3 + \frac{2}{3}} \lambda^3 \int e^{-i\lambda s_3 \Phi_2(u, z, s_2, \delta)} \chi_0(z) \chi_0(u) a(\sigma_\rho u, \rho^{2/3} z, s, \delta) \tilde{\chi}_1(s_2, s_3) du dz ds_2 ds_3,$$

with  $\Phi_2$  given by (10.10), and in place of (10.15) we now get

$$(11.7) \quad \|\nu_{\delta,0}^\lambda\|_\infty \lesssim \rho^{2/3} \lambda^2 (\lambda \rho)^{-1/3} = \rho^{-4/3} (\lambda \rho)^{5/3}.$$

By means of interpolation, we arrive from (11.4) and (11.4) at a uniform estimate

$$(11.8) \quad \|T_{\delta,0}^\lambda\|_{p_c \rightarrow p'_c} \lesssim 1.$$

The corresponding estimate for  $p < p_c$  allows for summation over all dyadic  $\lambda \gg 1$ , but in order to reach also the endpoint  $p = p_c$ , similar to our discussion in [21] for the case  $B = 2$ , we shall have to apply a complex interpolation argument.

For the estimation of the operators  $T_{\delta,l}^\lambda$ , very similar statements hold true (compare the analogous discussion in Subsection 10.2).

Scaling here  $x = \sigma_{2^l \rho} u := ((2^l \rho)^{1/3} u_1, (2^l \rho)^{1/3} u_2)$  leads to

$$(11.9) \quad \widehat{\nu}_\delta(\xi) = (2^l \rho)^{2/3} e^{-i\lambda s_3 B_0(s, \delta_1)} \int_{|\sigma_{2^l \rho} u| < \varepsilon} e^{-i\lambda 2^l \rho s_3 \Phi_1(u, s, \delta)} a(\sigma_{2^l \rho} u, \delta, s) du,$$

where now the phase  $\Phi_1 = \Phi_{1,l}$  has the form

$$(11.10) \quad \Phi_1(u, s, \delta) = u_1^3 B_3(s_2, \delta_1, (2^l \rho)^{1/3} u_1) - u_1 ((2^l \rho)^{-2/3} B_1(s, \delta_1)) + u_2^3 b(\sigma_{2^l \rho} u, \delta_0^r, s_2) \\ + \delta'_{3,0} u_2 \tilde{\alpha}_1(\delta_0^r, s_2) + \delta'_0 u_1 u_2 \alpha_{1,1}((2^l \rho)^{1/3} u_1, \delta_0^r, s_2),$$

and where

$$(\delta'_{3,0})^{\frac{3}{2}} + (\delta'_0)^3 = 2^{-l} \leq \frac{1}{M_0} \ll 1.$$

Moreover, in analogy with (11.3), one easily verifies that also

$$(11.11) \quad \sum_{\{l: M_0 \leq 2^l \leq \frac{p-1}{M_1}\}} \sum_{\lambda \rho \gg 1} \|T_{l,\infty}^\lambda\|_{p_c \rightarrow p'_c} \lesssim 1.$$

As for the main terms  $\nu_{l,0}^\lambda$  of

$$\nu_{\delta,l}^\lambda = \nu_{l,0}^\lambda + \nu_{l,\infty}^\lambda,$$

which is here given by

$$\widehat{\nu_{l,0}}(\xi) := (2^l \rho)^{\frac{2}{3}} e^{-i\lambda s_3 B_0(s, \delta_1)} \int_{|\sigma_{2^l \rho} u| < \varepsilon} e^{-i\lambda 2^l \rho s_3 \Phi_1(u, s, \delta)} a(\sigma_{2^l \rho} u, \delta, s) \chi_0(u) du,$$

we find that in place of (10.35) we now have

$$(11.12) \quad \|\widehat{\nu_{l,0}^\lambda}\|_\infty \lesssim (2^l \rho)^{\frac{2}{3}} (\lambda 2^l \rho)^{-\frac{5}{6}}.$$

Moreover, after changing coordinates, we may write

$$(11.13) \quad \begin{aligned} \nu_{l,0}^\lambda(x) &= (2^l \rho)^{\frac{2}{3} + \frac{2}{3}} \lambda^3 \\ &\times \int e^{-i\lambda s_3 \Phi_2(u, z, s_2, \delta)} \chi_1(z) \chi_0(u) a(\sigma_{2^l \rho} u, (2^l \rho)^{\frac{2}{3}} z, s, \delta) \tilde{\chi}_1(s_2, s_3) du dz ds_2 ds_3 \end{aligned}$$

with  $\Phi_2$  given by

$$(11.14) \quad \begin{aligned} \Phi_2(u, z, s_2, \delta) &:= s_2^{\frac{n}{n-2}} G_5(s_2, \delta) - x_1 s_2^{\frac{n-1}{n-2}} G_3(s_2, \delta) \\ &- s_2 x_2 - x_3 + (2^l \rho)^{2/3} z (x_1 - s_2^{\frac{1}{n-2}} G_1(s_2, \delta) + 2^l \rho \Phi_1(u, z, s_2, \delta)). \end{aligned}$$

and

$$(11.15) \quad \begin{aligned} \Phi_1(u, z, s_2, \delta) &= u_1^3 B_3(s_2, \delta_1, (2^l \rho)^{\frac{1}{3}} u_1) - u_1 z + u_2^3 b(\sigma_{2^l \rho} u, \delta_0^r, s_2) \\ &+ \delta'_{3,0} u_2 \tilde{\alpha}_1(\delta_0^r, s_2) + \delta'_0 u_1 u_2 \alpha_{1,1}((2^l \rho)^{\frac{1}{3}} u_1, \delta_0^r, s_2). \end{aligned}$$

This leads to the estimate

$$(11.16) \quad \|\nu_{l,0}^\lambda\|_\infty \lesssim (2^l \rho)^{\frac{2}{3}} \lambda^2 (\lambda 2^l \rho)^{-\frac{1}{3}}.$$

Interpolating between this estimate and (11.12), we obtain again only a uniform estimate

$$(11.17) \quad \|T_{l,0}^\lambda\|_{p_c \rightarrow p'_c} \lesssim 1$$

(whereas the corresponding estimate for  $p < p_c$  allows for summation over all dyadic  $\lambda \gg 1$  and all  $l$ .) So again, we shall have to apply a complex interpolation argument in order to include the endpoint  $p = p_c$ .

For the operators  $T_{l,\infty}^\lambda$ , we get a better estimate of the form  $\|T_{l,\infty}^\lambda\|_{p_c \rightarrow p'_c} \leq C_N (\lambda 2^l \rho)^{-N}$ , which allows to sum absolutely in  $l$  and  $\lambda$ .

Recall also that we have seen that we may restrict ourselves to those  $\lambda$  for which  $\lambda \ll \delta_0^{-3}$ . This assumption has the additional advantage that we shall have to deal only with finite sums in Proposition 11.3. Notice also that  $\rho^{-1} \leq \delta_0^{-3}$ , since we have seen that  $\rho \geq \delta_0^3$ .

Taking into account these observations, the following Proposition puts together those estimates which still need to be established in order to complete the proof of Proposition 2.1, and hence also that of our main result in [21], Theorem 1.7:

**Proposition 11.3.** *Assume that  $m = 2, B = 3$  and  $A = 0$  in (6.20), so that  $\theta_c = 1/3, p_c = 6/5$  and  $n \geq 7$ . Then the following hold true, provided  $M \in \mathbb{N}$  is sufficiently large and  $\delta$  sufficiently small:*

(a) *If  $\lambda\rho \lesssim 1$ , and if  $\nu_{\delta,Ai}^\lambda$  and  $\nu_{\delta,l}^\lambda$  are given by (8.23), respectively (8.24), then let*

$$\nu_{\delta,Ai}^I := \sum_{2^M \leq \lambda \leq 2^M \rho^{-1}} \nu_{\delta,Ai}^\lambda \quad \text{and} \quad \nu_\delta^{II} := \sum_{2^M \leq \lambda \leq 2^M \rho^{-1}} \sum_{\{l: M_0 \leq 2^l \leq \frac{\lambda}{M_1}\}} \nu_{\delta,l}^\lambda,$$

*and denote by  $T_{\delta,Ai}^I$  and  $T_\delta^{II}$  the convolution operators  $\varphi \mapsto \widehat{\varphi * \nu_{\delta,Ai}^I}$  and  $\varphi \mapsto \widehat{\varphi * \nu_\delta^{II}}$ , respectively. Then*

$$(11.18) \quad \|T_{\delta,Ai}^I\|_{p_c \rightarrow p'_c} \leq C \quad \text{and} \quad \|T_\delta^{II}\|_{p_c \rightarrow p'_c} \leq C.$$

(b) *If  $\lambda\rho \gg 1$ , and if  $\nu_{\delta,0}^\lambda$  and  $\nu_{l,0}^\lambda$  denote the main terms of  $\nu_{\delta,Ai}^\lambda$ , respectively  $\nu_{\delta,l}^\lambda$  (cf. (11.6), (11.13)), then let*

$$\nu_{\delta,Ai}^{III} := \sum_{2^M \rho^{-1} < \lambda \leq 2^{-M} \delta_0^{-3}} \nu_{\delta,0}^\lambda \quad \text{and} \quad \nu_\delta^{IV} := \sum_{\{l: M_0 \leq 2^l \leq \frac{\rho^{-1}}{M_1}\}} \sum_{2^M \rho^{-1} < \lambda \leq 2^{-M} \delta_0^{-3}} \nu_{l,0}^\lambda,$$

*and denote by  $T_{\delta,Ai}^{III}$  and  $T_\delta^{IV}$  the convolution operators  $\varphi \mapsto \widehat{\varphi * \nu_{\delta,Ai}^{III}}$  and  $\varphi \mapsto \widehat{\varphi * \nu_\delta^{IV}}$ , respectively. Then*

$$(11.19) \quad \|T_{\delta,Ai}^{III}\|_{p_c \rightarrow p'_c} \leq C \quad \text{and} \quad \|T_\delta^{IV}\|_{p_c \rightarrow p'_c} \leq C.$$

*Here, the constant  $C$  is independent of  $\delta$ .*

## 12. PROOF OF PROPOSITION 11.3 (A) : COMPLEX INTERPOLATION

In this section, we assume that  $B = 3$  and  $A = 0$  in (6.20), and that  $\lambda\rho \lesssim 1$ .

12.1. **Estimation of  $T_{\delta, Ai}^I$ .** Recall formula (8.30) for  $\widehat{\nu_{\delta, Ai}^\lambda}$ . Applying the Fourier inversion formula to this expression and performing the change of variables  $z := \lambda^{\frac{2}{3}} B_1(s, \delta)$ , we find that we may write

$$(12.1) \quad \nu_{\delta, Ai}^\lambda(x) = \lambda^{\frac{5}{3}} \int e^{-is_3 \lambda \Phi(z, s_2, x, \delta)} a(z, s_2, \tilde{\delta}^\lambda, \delta_0^r, \lambda^{-\frac{1}{9}}) \chi_0(z) \tilde{\chi}_1(s_2) \chi_1(s_3) dz ds_2 ds_3,$$

where  $a$  is again a smooth function of all its (bounded) variables, and where

$$\Phi(z, s_2, x, \delta) := \phi(s_2, x, \delta) + \lambda^{-\frac{2}{3}} z(x_1 - s_2^{\frac{1}{n-2}} G_1(s_2, \delta)),$$

with

$$\phi(s_2, x, \delta) := s_2^{\frac{n}{n-2}} G_5(s_2, \delta) - s_2^{\frac{n-1}{n-2}} G_3(s_2, \delta) x_1 - s_2 x_2 - x_3$$

(compare with a similar discussion in Section 6 of [21], in particular with (6.22)). Recall also that  $G_3(0) \neq 0$  and  $G_5(0) = (G_1 G_3 - G_2)(0) \neq 0$ .

In fact, a priori we have to assume that the density  $a$  depends also on the variable  $s_3$ . However, arguing as in Subsection 10.1, we may develop this function into a convergent series of smooth functions, each of which is a tensor product of a smooth function of the variable  $s_3$  with a smooth function depending on the remaining variables only. Thus, by considering each of the corresponding terms separately, we can reduce to the situation (12.1), provided we choose the functions  $\tilde{\chi}_1$  and  $\chi_1$ , which localize to the regions where  $|s_2| \sim 1$  and  $|s_3| \sim 1$ , properly.

We next embed  $\nu_{\delta, Ai}^I$  into an analytic family of measures

$$\nu_{\delta, \zeta}^I := \gamma(\zeta) \sum_{2^M \leq \lambda \leq 2^{M+1}} \lambda^{\frac{2}{3}(1-3\zeta)} \nu_{\delta, Ai}^\lambda$$

where  $\zeta$  lies again in the complex strip  $\Sigma$  given by  $0 \leq \operatorname{Re} \zeta \leq 1$ , and where  $\gamma(\zeta) := (1 - 2^{2(1-\zeta)})/(1 - 2^{4/3})$ .

Here, summation is again over dyadic  $\lambda = 2^j, j \in \mathbb{N}$ . Observe that indeed  $\nu_{\delta, Ai}^I = \nu_{\delta, \theta_c}^I$ , since  $\theta_c = 1/3$ .

Since the supports of the  $\widehat{\nu_{\delta, Ai}^\lambda}$  are almost disjoint, (8.25) implies that

$$\|\widehat{\nu_{\delta, it}^I}\|_\infty \lesssim 1 \quad \forall t \in \mathbb{R}.$$

We shall also prove that

$$(12.2) \quad |\nu_{\delta, 1+it}^I(x)| \leq C \quad \forall t \in \mathbb{R}, x \in \mathbb{R}^3.$$

Again, Stein's interpolation theorem will then imply that the operator  $T_{\delta, Ai}^I$  is bounded from  $L^{p_c}$  to  $L^{p'_c}$ , which will complete the first part of Proposition 11.3 (a).

In order to prove (12.2), we first consider the case where  $|x| \gg 1$ . In this case, we can use a formula similar to (6.18) in [21] in order to argue as in Subsection 6.1 of [21] and find that

$$|\nu_{\delta, Ai}^\lambda(x)| \leq C_N \lambda^{-N}, \quad N \in \mathbb{N}, \text{ if } |x| \gg 1.$$

This estimate allows in a trivial way to sum in  $\lambda$  and thus to obtain (12.2).

We may therefore assume from now on that  $|x| \lesssim 1$ . We then write

$$(12.3) \quad \nu_{\delta, 1+it}^I(x) = \gamma(1+it) \sum_{2^M \leq \lambda \leq 2^{M+p-1}} \lambda^{-2it} \mu_\lambda(x),$$

where

$$\mu_\lambda(x) := \lambda^{\frac{1}{3}} \int e^{-is_3 \lambda \Phi(z, s_2, x, \delta)} a(z, s_2, \tilde{\delta}^\lambda, \delta_0^\tau, \lambda^{-\frac{1}{9}}) \chi_0(z) \tilde{\chi}_1(s_2) \chi_1(s_3) dz ds_2 ds_3.$$

Let us look at the contribution to this integral given by a small neighborhood of a given point  $s_2^0 \sim 1$ . If  $|\partial_{s_2}^2 \phi(s_2^0, x, \delta)| \sim 1$ , then van der Corput's estimate applied to the integration in  $s_2$  shows that  $|\mu_\lambda(x)| \lesssim \lambda^{1/3} \lambda^{-1/2} = \lambda^{-1/6}$ , which clearly implies (12.2). This situation arises in particular when  $|x_1| \ll 1$ , or when  $|x_1| \sim 1$  and  $G_5$  and  $G_3 x_1$  have opposite signs.

So, let us assume from now on that  $|x_1| \sim 1$ , and that  $G_5$  and  $x_1 G_3$  have the same sign. Notice that if  $\delta = 0$ , then

$$\phi(s_2, x, 0) = s_2^{\frac{n}{n-2}} G_5(0) - s_2^{\frac{n-1}{n-2}} G_3(0) x_1 - s_2 x_2 - x_3.$$

Since the exponents  $n/(n-1)$  and  $(n-1)/(n-2)$  are different, this shows that there is a unique  $s_2^c(0) \sim 1$  so that  $\partial_{s_2}^2 \phi(s_2^c(0), x, 0) = 0$ , whereas  $|\partial_{s_2}^3 \phi(s_2^c(0), x, 0)| \sim 1$ . By the implicit function, we then find a smooth function  $s_2^c(x_1, \delta)$  such that

$$\partial_{s_2}^2 \phi(s_2^c(x_1, \delta), x, \delta) \equiv 0.$$

Let us put here

$$\phi^\sharp(v, x, \delta) := \phi(s_2^c(x_1, \delta) + v, x, \delta), \quad \Phi^\sharp(z, v, x, \delta) := \phi(z, s_2^c(x_1, \delta) + v, x, \delta).$$

By means of Taylor expansion around  $v = 0$ , we may write

$$\phi^\sharp(v, x, \delta) = v^3 Q_3(v, x, \delta) - v Q_1(x, \delta) + Q_0(x, \delta),$$

where the  $Q_j$  are smooth functions of all their variables, and where we may assume that  $|Q_3(v, x, \delta)| \sim 1$ , since we had  $|\partial_{s_2}^3 \phi(s_2^c(0), x, 0)| \sim 1$ . Moreover, developing

$$x_1 - (s_2^c(x_1, \delta) + v)^{\frac{1}{n_2-2}} G_1(s_2^c(x_1, \delta) + v, \delta_1) = Q_5(x, \delta) + v Q_6(v, x, \delta),$$

after scaling  $v \mapsto \lambda^{-1/3} v$ , we find that

$$\begin{aligned} \lambda \Phi^\sharp(z, \lambda^{-\frac{1}{3}} v, x, \delta) &= v^3 Q_3(\lambda^{-\frac{1}{3}} v, x, \delta) - v \lambda^{\frac{2}{3}} Q_1(x, \delta) + \lambda Q_0(x, \delta) \\ &\quad + z v Q_6(\lambda^{-\frac{1}{3}} v, x, \delta) + \lambda^{\frac{1}{3}} z Q_5(x, \delta). \end{aligned}$$

This allows to re-write

$$(12.4) \quad \mu_\lambda(x) = \int \int_{-\lambda^{1/3}}^{\lambda^{1/3}} \chi_0(z) F_\delta(\lambda, x, z, v) dv dz,$$



where  $F_\delta$  is of the form

$$F_\delta(\lambda, x, z, v) = \chi_0(\lambda^{-\frac{1}{3}}v) \widehat{\chi_1} \left( A + Dz - Bv - v^3 Q_3(\lambda^{-\frac{1}{3}}v, x, \delta) + zv Q_6(\lambda^{-\frac{1}{3}}v, x, \delta) \right) \\ \tilde{a}(z, \lambda^{-\frac{1}{3}}v, \tilde{\delta}^\lambda, x_1, \delta_0^\sharp, \lambda^{-\frac{1}{9}}).$$

Here,  $\tilde{a}$  is again a smooth function of all its variables with compact support, and

$$A = A(x, \lambda, \delta) := \lambda Q_0(x, \delta), \quad B := B(x, \lambda, \delta) := \lambda^{\frac{2}{3}} Q_1(x, \delta), \\ D = D(x, \lambda, \delta) := \lambda^{\frac{1}{3}} Q_5(x, \delta).$$

Now we can argue in a very similar way as in previous proofs, and will therefore briefly sketch the proof.

We first consider the contribution to the sum in (12.3) given by the  $\lambda$ 's satisfying  $|A| = \lambda |Q_0(x, \delta)| \gg 1$ . Observe that for  $z$  fixed, we may estimate  $\int_{-\lambda^{1/3}}^{\lambda^{1/3}} |F_\delta(\lambda, x, z, v)| dv$  by means of Lemma 14.1, where we choose  $y_2 = v$ ,  $T := \lambda^{1/3}$ ,  $\epsilon = 0$ ,  $r_i \equiv 0$  (so that the integral in  $y_1$  just yields a positive constant) and  $Q(y_2) = z Q_6(y_2, x, \delta)$ . The condition (14.1) is here also satisfied, since  $(\phi^\sharp)''(0, x, \delta) = 0$  and  $|(\phi^\sharp)'''(0, x, \delta)| \sim 1$ . Thus, Lemma 14.1 implies that for  $L$  sufficiently large, we may estimate

$$|\mu_\lambda(x)| \leq C \left( \int_I |A + Dz|^{-\frac{1}{6}} |\chi_0(z)| dz + |J| \right),$$

where  $I$  and  $J$  denote the sets of all  $z \in \text{supp } \chi_0$  for which  $|A + Dz| \geq L$ , respectively  $|A + Dz| < L$ . The integral can easily be estimated by

$$|D|^{-\frac{1}{6}} \int_I \left| z + \frac{A}{D} \right|^{-\frac{1}{6}} |\chi_0(z)| dz \lesssim |D|^{-\frac{1}{6}} (1 + \left| \frac{A}{D} \right|)^{-\frac{1}{6}} \lesssim |A|^{-\frac{1}{6}}.$$

Moreover, if  $|D| \ll L$ , then the set  $J$  is empty, if we assume that  $|A| \geq 2L$ . So, let us assume that  $|D| \gtrsim L$ . Then, on the set  $J$  we have  $|z + A/D| < L/|D|$ , and since  $|z| \lesssim 1$ , this implies that  $|A/D| \lesssim 1$ , and we see that  $|J| \leq L/|D| \lesssim L/|A|$ . Putting all this together, we find that

$$|\mu_\lambda(x)| \leq C |A|^{-\frac{1}{6}},$$

which allows to sum over those  $\lambda$  for which  $\lambda |Q_0(x, \delta)| \gg 1$  so that the estimate is independent of  $x$ .

Next, we consider the  $\lambda$ 's for which  $|A| \lesssim 1$ . If in addition  $|D| \gg 1$ , then we can argue as before and obtain an estimate of the form  $|\mu_\lambda(x)| \leq C |D|^{-1/6}$ , which again allows to sum. Similarly, if  $|A| \lesssim 1$  and  $|D| \lesssim 1$ , but  $|B| \gg 1$ , we may apply Lemma 14.1 once more and obtain that  $|\mu_\lambda(x)| \leq C |B|^{-1/4}$ . This allows again to sum.

We are thus left with the oscillatory sum (12.3) over only those  $\lambda$ 's for which, say,  $\max\{|A|, |B|, |D|\} \leq L$ . However, this can again easily be handled by means of Lemma 5.2, and we arrive at (12.2). Recall here that by (8.19) the components of  $\tilde{\delta}^\lambda$  are of the form  $\lambda^{\beta_i} \delta_i$ , where we may assume that  $|\lambda^{\beta_i} \delta_i| \leq C$ , since we are assuming that  $\rho(\tilde{\delta}^\lambda) = \lambda \rho(\tilde{\delta}) \lesssim 1$ .

**12.2. Estimation of  $T_\delta^{II}$ .** We next come to the proof the second estimate in Proposition 11.3 (a), where we still assume that  $\lambda\rho \lesssim 1$ . Recall from Subsection 8.2 that we have decomposed

$$(12.5) \quad \nu_{\delta,l}^\lambda = \nu_{l,\infty}^\lambda + \nu_{l,0,0}^\lambda + \nu_{l,I}^\lambda + \nu_{l,II}^\lambda + \nu_{l,III}^\lambda.$$

Here, we denote by  $\nu_{l,0,0}^\lambda$  the contribution to  $\nu_{l,0}^\lambda$  by the domain where  $|u_1| \ll 1$ .

Again, we embed the measure  $\nu_\delta^{II}$  into an analytic family of measures

$$\nu_{\delta,\zeta}^{II} := \gamma(\zeta) \sum_{2^M \leq \lambda \leq 2^M \rho^{-1}} \sum_{\{l: M_0 \leq 2^l \leq \frac{\lambda}{M_1}\}} 2^{\frac{l}{6}(1-3\zeta)} \lambda^{\frac{2}{3}(1-3\zeta)} \nu_{\delta,l}^\lambda$$

where  $\zeta$  lies again in the complex strip  $\Sigma$ . The analytic function  $\gamma(\zeta)$  will be a finite product of factors  $\gamma_i(\zeta)$  which will be specified in the course of the proof.

In view of (8.27), by following our standard approach it will suffice to prove the following estimate in order to establish the second inequality in (11.18):

$$(12.6) \quad |\nu_{\delta,1+it}^{II}(x)| \leq C \quad \forall t \in \mathbb{R}, x \in \mathbb{R}^3.$$

By putting  $\mu_{l,\lambda} := 2^{-\frac{1}{3}l} \lambda^{-\frac{4}{3}} \nu_{\delta,l}^\lambda$ , we may re-write

$$(12.7) \quad \nu_{\delta,1+it}^{II}(x) = \gamma(1+it) \sum_{2^M \leq \lambda \leq 2^M \rho^{-1}} \sum_{\{l: M_0 \leq 2^l \leq \frac{\lambda}{M_1}\}} 2^{-\frac{1}{2}itl} \lambda^{-2it} \mu_{l,\lambda}(x).$$

**12.2.1. Contribution by the  $\nu_{l,II}^\lambda$ .** Let us begin with the contribution of the main terms  $\nu_{l,II}^\lambda$  in (12.5), i.e., let us look at

$$(12.8) \quad \nu_{II,1+it} := \gamma(1+it) \sum_{2^M \leq \lambda \leq 2^M \rho^{-1}} \sum_{\{l: M_0 \leq 2^l \leq \frac{\lambda}{M_1}\}} 2^{-\frac{1}{2}itl} \lambda^{-2it} \mu_{l,\lambda,II}(x),$$

where we have set  $\mu_{l,\lambda,II} := 2^{-\frac{1}{3}l} \lambda^{-\frac{4}{3}} \nu_{l,II}^\lambda$ . We want to prove that

$$(12.9) \quad |\nu_{II,1+it}(x)| \leq C \quad \forall t \in \mathbb{R}, x \in \mathbb{R}^3.$$

Recall from Subsection 8.2 that we may restrict ourselves to those  $x$  for which  $|x| \lesssim 1$  and  $|x_1| \sim 1$ . Making use of (8.53) and (8.54), after scaling the variable  $u_2$  by the factor  $2^{-l/3}$ , we find that

$$(12.10) \quad \begin{aligned} \mu_{l,\lambda,II}(x) &= \int e^{-is_3 \tilde{\Psi}_3(y_2, v, x, \delta, \lambda, l)} a_3(\lambda^{-\frac{1}{3}} y_2, v, x, 2^{-\frac{l}{3}}, (2^l \lambda^{-1})^{\frac{1}{3}}, \delta) \\ &\quad \times \chi_1(s_3) \chi_1(v) \chi_0(2^{-\frac{l}{3}} y_2) dy_2 dv ds_3 \\ &= \int \widehat{\chi}_1(\tilde{\Psi}_3(y_2, v, x, \delta, \lambda, l)) a_3(\lambda^{-\frac{1}{3}} y_2, v, x, 2^{-\frac{l}{3}}, (2^l \lambda^{-1})^{\frac{1}{3}}, \delta) \chi_1(v) \chi_0(2^{-\frac{l}{3}} y_2) dy_2 dv, \end{aligned}$$

with

$$\begin{aligned}\tilde{\Psi}_3(y_2, v, x, \delta, \lambda, l) &= v \lambda (2^{-l} \lambda)^{-\frac{1}{3}} (x_1^2 \omega(\delta_1 x_1) - x_2) + \lambda Q_A(x, \delta) \\ &\quad + y_2^3 b(x_1, \lambda^{-\frac{1}{3}} y_2, \delta) + y_2 \left( \lambda^{\frac{2}{3}} [\delta_0 \tilde{G}_1(x_1, \delta) + \delta_3 x_1^{n_1} \alpha_1(\delta_1 x_1)] + (2^l \lambda)^{\frac{1}{3}} \delta_0 v \right).\end{aligned}$$

We shall write this as

$$(12.11) \quad \tilde{\Psi}_3(y_2, v, x, \delta, \lambda, l) = A + Bv + y_2^3 b(x_1, \lambda^{-\frac{1}{3}} y_2, \delta) + y_2(D + Ev),$$

with

$$(12.12) \quad \begin{aligned}A &= A(x, \lambda, \delta) := \lambda Q_A(x, \delta), & B &= B(x, \lambda, l, \delta) := 2^{\frac{l}{3}} \lambda^{\frac{2}{3}} Q_B(x, \delta), \\ D &= D(x, \lambda, \delta) := \lambda^{\frac{2}{3}} Q_D(x, \delta), & E &= E(\lambda, l, \delta) = 2^{\frac{l}{3}} \lambda^{\frac{1}{3}} \delta_0,\end{aligned}$$

and

$$\begin{aligned}Q_A(x, \delta) &:= \tilde{G}_1(x_1, \delta) (x_1^2 \omega(\delta_1 x_1) - x_2) + x_1^n \alpha(\delta_1 x_1) - x_3, \\ Q_B(x, \delta) &:= x_1^2 \omega(\delta_1 x_1) - x_2, & Q_D(x, \delta) &:= \delta_0 \tilde{G}_1(x_1, \delta) + \delta_3 x_1^{n_1} \alpha_1(\delta_1 x_1).\end{aligned}$$

Now, applying Lemma 14.2 from the appendix to the integration in  $y_2$ , with  $T := 2^{l/3}$  and  $\delta := \lambda^{-1/3}$  (so that  $\delta T = (2^l \lambda^{-1})^{1/3} \ll 1$ ), we see that we may estimate

$$(12.13) \quad \begin{aligned}& \left| \int \widehat{\chi}_1(\tilde{\Psi}_3(y_2, v, x, \delta, \lambda, l)) a_3(\lambda^{-\frac{1}{3}} y_2, v, x, 2^{-\frac{l}{3}}, (2^l \lambda^{-1})^{\frac{1}{3}}, \delta) \chi_0(2^{-\frac{l}{3}} y_2) dy_2 \right| \\ & \lesssim (1 + \max\{|A + Bv|, |D + Ev|\})^{-\frac{1}{6}},\end{aligned}$$

with a constant which does not depend on  $A, B, D, E$  and  $v$ . The following simple lemma will thus be useful:

**Lemma 12.1.** *Let  $\varepsilon > 0$ , and consider for  $A, B, D, E \in \mathbb{R}$  the integral*

$$J(A, B, D, E) := \int (1 + \max\{|A + Bv|, |D + Ev|\})^{-\varepsilon} \chi_0(v) dv,$$

where as usually  $\chi_0$  is a smooth, non-negative bump function with compact support. Then

$$|J(A, B, D, E)| \leq C (\max\{|A|, |B|, |D|, |E|\})^{-\varepsilon},$$

where the constant  $C$  is independent of  $A, B, D$  and  $E$ .

*Proof.* If  $|A| \gg |B|$ , then  $|A + Bv| \gtrsim |A|$ , and we see that  $|J(A, B, D, E)| \leq C|A|^{-\varepsilon}$ . Next, if  $|A| \lesssim |B|$  then we apply the change of variables  $v \mapsto w := Bv$  and find that we can estimate

$$|J(A, B, D, E)| \lesssim \frac{1}{|B|} \int_{|w| \lesssim |B|} (1 + |A + w|)^{-\varepsilon} dw \lesssim \frac{1}{|B|} \int_{|y| \lesssim |B|} (1 + |y|)^{-\varepsilon} dy \lesssim |B|^{-\varepsilon},$$

provided  $|B| \geq 1$ . Of course, if  $|B| < 1$ , then we can always use the trivial estimate  $|J(A, B, D, E)| \lesssim 1$ .

We may now conclude the proof by interchanging the roles of  $A, B$  and  $D, E$ .

Q.E.D.

In combination with (12.13) and (12.10) this lemma implies that

$$(12.14) \quad |\mu_{l,\lambda,II}(x)| \lesssim (\max\{|A|, |B|, |D|, |E|\})^{-\frac{1}{6}},$$

with  $A, B, D$  and  $E$  given by (12.12).

In order to estimate  $\nu_{II,1+it}(x)$ , we shall again distinguish various cases, in a similar way as in preceding arguments of this type, and shall thus only briefly sketch the ideas.

Let us first consider the contribution by those terms in (12.8) for which  $|D| \gtrsim 1$  and  $|E| \gtrsim 1$ . Since by (12.14) we may estimate  $|\mu_{l,\lambda,II}(x)| \lesssim |D|^{-1/12} |E|^{-1/12}$ , we can first sum these contributions absolutely over all  $l$  for which  $|E| = 2^{l/3} \lambda^{1/3} \delta_0 \gtrsim 1$ , and subsequently over all dyadic  $\lambda = 2^j$  for which  $|D| = \lambda^{\frac{2}{3}} |Q_D(x, \delta)| \gtrsim 1$ , and arrive at a bound which is uniform in  $x$  and  $\delta$ .

In essentially the same way we can sum (absolutely) the contributions by those terms in (12.8) for which  $|A| \gtrsim 1$  and  $|B| \gtrsim 1$ .

Consider next the terms for which  $|E| \ll 1$  and  $|D| \ll 1$ . For these terms, we re-write

$$(12.15) \quad \mu_{l,\lambda,II}(x) = \int e^{-is_3(A+Bv)} J(v, s_3) \chi_1(s_3) \chi_1(v) dv ds_3$$

with

$$(12.16) \quad \begin{aligned} J(v, s_3) := & \int e^{-is_3 \left( y_2^3 b(x_1, \lambda^{-\frac{1}{3}} y_2, \delta) + y_2(D+Ev) \right)} a_3(\lambda^{-\frac{1}{3}} y_2, v, x, 2^{-\frac{l}{3}}, (2^l \lambda^{-1})^{\frac{1}{3}}, \delta) \\ & \times \chi_0(2^{-\frac{l}{3}} y_2) dy_2, \end{aligned}$$

But, since  $|D + Ev| \lesssim 1$ , the proof of Lemma 6.3 (a) in [21] shows that  $J(v, s_3) = g(D + Ev, s_3, x, 2^{-\frac{l}{3}}, (2^l \lambda^{-1})^{\frac{1}{3}}, \delta)$ , with a smooth function  $g$ , and thus

$$\mu_{l,\lambda,II}(x) = \int e^{-is_3(A+Bv)} g(D + Ev, s_3, x, 2^{-\frac{l}{3}}, (2^l \lambda^{-1})^{\frac{1}{3}}, \delta) \chi_1(s_3) \chi_1(v) dv ds_3.$$

Arguing in a similar way as in Subsection 12.1, without loss of generality we may and shall assume that  $g(D + Ev, s_3, x, 2^{-\frac{l}{3}}, (2^l \lambda^{-1})^{\frac{1}{3}}, \delta) = g(D + Ev, x, 2^{-\frac{l}{3}}, (2^l \lambda^{-1})^{\frac{1}{3}}, \delta)$  is independent of  $s_3$ . Then we may write

$$(12.17) \quad \mu_{l,\lambda,II}(x) = \int g(D + Ev, x, 2^{-\frac{l}{3}}, (2^l \lambda^{-1})^{\frac{1}{3}}, \delta) \widehat{\chi_1}(A + Bv) \chi_1(v) dv$$

(alternatively, one could also use integrations by parts in  $s_3$  in the previous formula, but the other approach appears a bit clearer).

Recall that we assume that either  $|A| \gtrsim 1$  and  $|B| \ll 1$ , or  $|A| \ll 1$  and  $|B| \gtrsim 1$ , or  $|A| \ll 1$  and  $|B| \ll 1$ .

If  $|A| \gtrsim 1$  and  $|B| \ll 1$ , then we can treat the summation in  $l$  by means of Lemma 5.2, where we choose, for  $\lambda$  fixed,

$$H_{\lambda,x}(u_1, u_2, u_3, u_4) := \int g(D + u_1 v, x, u_3, u_4, \delta) \widehat{\chi_1}(A + u_2 v) \chi_1(v) dv.$$

Then clearly  $\|H_{\lambda,x}\|_{C^1(Q)} \lesssim |A|^{-1}$ , and so after summation in those  $l$  for which  $|E| \ll 1$  and  $|B| \ll 1$ , we can also sum (absolutely) in the  $\lambda$ 's for which  $|A| \gtrsim 1$ . Observe that this requires that  $\gamma(\zeta)$  contains a factor

$$\gamma_1(\zeta) := \frac{2^{\frac{1-\zeta}{2}} - 1}{2^{\frac{1}{3}} - 1}.$$

Consider next the case where  $|B| \gtrsim 1$  and  $|A| \ll 1$ . If we write  $\lambda = 2^j$ , then  $2^{l/3} \lambda^{2/3} = 2^{k/3}$ , where we put  $k := l + 2j$ . We therefore pass from the summation variables  $j$  and  $l$  to the variables  $j$  and  $k$ , which allows to write  $B = 2^{k/3} Q_B(x)$ . For  $k$  fixed, we then sum first in  $j$  by means of Lemma 5.2, which gives an estimate of order  $O(|B|^{-1})$ , which then in return allows to sum (absolutely) in those  $k$  for which  $|B| = 2^{k/3} |Q_B(x)| \gtrsim 1$ . Since  $-l/2 - 2j = -k/2 - j$ , the application of Lemma 5.2 requires in this case that  $\gamma(\zeta)$  contains a factor

$$\gamma_2(\zeta) := \frac{2^{1-\zeta} - 1}{2^{\frac{2}{3}} - 1}.$$

There remains the case where  $|A| + |B| \ll 1$  and  $|D| + |E| \lesssim 1$ . The summation over all  $l$ 's and  $\lambda$ 's for which these conditions are satisfied can easily be treated by means of the double summation Lemma 8.1 in [21], in a very similar way as this was done in the last part of the proof of Proposition 5.2 (a) of that article. The corresponding vector  $(\alpha_1, \alpha_2)$  to be used in Lemma 8.1 will here be given by  $(\alpha_1, \alpha_2) = (2, 1/2)$ , and the vectors  $(\beta_1^k, \beta_2^k)$  by  $(1, 0), (2/3, 1/3), (2/3, 0), (1/3, 1/3), (-1, 1)$  and  $(0, -1/3)$ ; they obviously satisfy the assumptions of Lemma 8.1. For the application of this lemma, we need to assume that  $\gamma(\zeta)$  contains also a factor  $\gamma_3(\zeta)$  given by Remark 8.2 in [21].

What remains are the contributions by those  $l$  and  $\lambda$  for which either  $|D| \gtrsim 1$  and  $|E| \ll 1$ , or  $|D| \ll 1$  and  $|E| \gtrsim 1$ .

We begin with the case where  $|E| \gtrsim 1$  and  $|D| \ll 1$ . Then we may assume in addition that  $|B| \ll 1$ , for otherwise by (12.14) we have  $|\mu_{l,\lambda,II}(x)| \lesssim |E|^{-1/12} |B|^{-1/12}$ , which allows to sum absolutely in  $j$  and  $l$ , as can easily seen by means of a change of the summation variables from  $j$  and  $l$  to  $k := 2j + l$  and  $m := j + l$  (compare (12.12)). In a very similar way, we may also assume that  $|A| \ll 1$ .

Recall from (12.10) and (12.11) that

$$\begin{aligned} \mu_{l,\lambda,II}(x) = & \int e^{-is_3(A+y_2^3 b(x_1, \lambda^{-\frac{1}{3}} y_2, \delta) + D y_2 + v(B+E y_2))} a_3(\lambda^{-\frac{1}{3}} y_2, v, x, 2^{-\frac{l}{3}}, (2^l \lambda^{-1})^{\frac{1}{3}}, \delta) \\ & \times \chi_1(s_3) \chi_1(v) \chi_0(2^{-\frac{l}{3}} y_2) dy_2 dv ds_3. \end{aligned}$$

Again, by our usual argument, we may assume without loss of generality that  $a_3$  is independent of  $v$ . Then find that

$$(12.18) \quad \begin{aligned} \mu_{l,\lambda,II}(x) &= \int e^{-is_3(A+y_2^3 b(x_1, \lambda^{-\frac{1}{3}} y_2, \delta))} \widehat{\chi_1}(s_3(B + Ey_2)) \\ &\times a_3(\lambda^{-\frac{1}{3}} y_2, x, 2^{-\frac{l}{3}}, (2^l \lambda^{-1})^{\frac{1}{3}}, \delta) \chi_1(s_3) \chi_1(v) \chi_0(2^{-\frac{l}{3}} y_2) dy_2 ds_3. \end{aligned}$$

We then change the summation variables from  $j, l$  to  $j, k$ , where  $k := j + l$ , so that  $E = 2^{k/3} \delta_0$ . Then, for  $k$  fixed, we can treat the summation in  $j$  by means of Lemma 5.2, where we choose

$$\begin{aligned} H_{k,x}(u_1, u_2, u_3, u_4, u_5) &:= \int e^{-is_3(u_1+y_2^3 b(x_1, u_2 y_2, \delta))} \widehat{\chi_1}(s_3(u_3 + Ey_2)) \\ &\times a_3(u_2 y_2, x, u_4, u_5, \delta) \chi_1(s_3) \chi_1(v) \chi_0(u_3 y_2) dy_2 ds_3. \end{aligned}$$

By means of the change of variables  $y_2 \mapsto y_2/E$  we thus see that  $|H_{k,x}(u_1, \dots, u_5)| \lesssim |E|^{-1}$  on the natural cuboid  $Q$  arising in this context, since  $\widehat{\chi_1}$  is a Schwartz function. Next, observe that by the product rule,  $\partial_{u_i} H_{k,x}(u_1, \dots, u_5)$  can be written as finite sum of integrals of a similar form, where the amplitude may carry additional factors of the form  $y_2^n$ , with  $n = 0, \dots, 4$ . Again, the change of variables  $y_2 \mapsto y_2/E$  shows that these can be estimated by  $C|E|^{-1}$  (or even higher powers of  $|E|^{-1}$ ). We thus find that  $\|H_{k,x}\|_{C^1(Q)} \lesssim |E|^{-1}$ , and thus after the summation over the  $\lambda = 2^j$  this allows to subsequently also sum over the  $k$  for which  $|E| = 2^{k/3} \delta_0 \gtrsim 1$ .

There remains the contribution by those  $l$  and  $\lambda$  for which  $|D| \gtrsim 1$  and  $|E| \ll 1$ . Observe that here we have  $|D + Ev| \gtrsim 1$  in (12.16).

Applying the change of variables  $y_2 = \lambda^{1/3} t$  in the integral defining  $J(v, s_3)$ , we obtain

$$(12.19) \quad \begin{aligned} J(v, s_3) &= \lambda^{\frac{1}{3}} \int e^{-is_3 \lambda (t^3 b(x_1, t, \delta) + t(\lambda^{-\frac{2}{3}}(D + Ev)))} \\ &\times a_3(t, v, x, 2^{-\frac{l}{3}}, (2^l \lambda^{-1})^{\frac{1}{3}}, \delta) \chi_0((\lambda 2^{-l})^{\frac{1}{3}} t) dt. \end{aligned}$$

It is important to observe that here the phase function is independent of  $l$ . Notice also that by (12.12)

$$(12.20) \quad \lambda^{-\frac{2}{3}} D = Q_D(x, \delta), \quad \lambda^{-\frac{2}{3}} E = (2^l \lambda^{-1})^{\frac{1}{3}} \delta_0 \ll 1,$$

so that in particular  $\lambda^{-\frac{2}{3}} |D + vE| \lesssim 1$ .

We may then argue in a similar way as in the proof of Lemma 6.3 (b) in [21] to see that for every  $N \in \mathbb{N}$ ,

$$\begin{aligned}
 (12.21) \quad J(v, s_3) &= |D + Ev|^{-\frac{1}{4}} a_+ \left( \lambda^{-\frac{1}{3}} |D + Ev|^{\frac{1}{2}}, v, x, 2^{-\frac{l}{3}}, (2^l \lambda^{-1})^{\frac{1}{3}}, \delta \right) \\
 &\quad \times \chi_0 \left( 2^{-\frac{l}{3}} |D + Ev|^{\frac{1}{2}} \right) e^{-is_3 |D + Ev|^{\frac{3}{2}} q_+ \left( \lambda^{-\frac{1}{3}} |D + Ev|^{\frac{1}{2}}, x, \delta \right)} \\
 &+ |D + Ev|^{-\frac{1}{4}} a_- \left( \lambda^{-\frac{1}{3}} |D + Ev|^{\frac{1}{2}}, v, x, 2^{-\frac{l}{3}}, (2^l \lambda^{-1})^{\frac{1}{3}}, \delta \right) \\
 &\quad \times \chi_0 \left( 2^{-\frac{l}{3}} |D + Ev|^{\frac{1}{2}} \right) e^{-is_3 |D + Ev|^{\frac{3}{2}} q_- \left( \lambda^{-\frac{1}{3}} |D + Ev|^{\frac{1}{2}}, x, \delta \right)} \\
 &+ (D + Ev)^{-N} F_N \left( |D + vE|^{\frac{3}{2}}, \lambda^{-\frac{1}{3}} |D + Ev|^{\frac{1}{2}}, v, x, 2^{-\frac{l}{3}}, (2^l \lambda^{-1})^{\frac{1}{3}}, \delta \right),
 \end{aligned}$$

where  $a_{\pm}, q_{\pm}$  and  $F_N$  are smooth functions of their (bounded) variables. Moreover,  $|q_{\pm}((0, x, (2^l \lambda^{-1})^{\frac{1}{3}}, \delta))| \sim 1$ .

Indeed, notice that the phase in (12.19) has a critical point in the support of the amplitude only if  $2^{-\frac{l}{3}} |D + vE|^{1/2} \lesssim 1$ , and so we obtain the first two terms in (12.21) by applying the method of stationary phase, whereas the last term arises from integrations by parts on intervals in  $t$  on which there is no stationary point.

We shall concentrate on the first term only. The second term can be treated in the same way as the first one, and the last term can be handled in an even easier way by a similar method, since it is of order  $= O(|D|^{-N})$  and, unlike the first term, carries no oscillatory factor.

We denote by

$$\begin{aligned}
 \mu_{l,\lambda}^1(x) &:= \int e^{-is_3 \left[ (A+Bv) + |D + Ev|^{\frac{3}{2}} q_+ \left( \lambda^{-\frac{1}{3}} |D + Ev|^{\frac{1}{2}}, x, \delta \right) \right]} |D + Ev|^{-\frac{1}{4}} \chi_1(s_3) \chi_1(v) \\
 &\quad \times a_+ \left( \lambda^{-\frac{1}{3}} |D + Ev|^{\frac{1}{2}}, v, x, 2^{-\frac{l}{3}}, (2^l \lambda^{-1})^{\frac{1}{3}}, \delta \right) \chi_0 \left( 2^{-\frac{l}{3}} |D + Ev|^{\frac{1}{2}} \right) dv ds_3
 \end{aligned}$$

the contribution by the first term in (12.21) to  $\mu_{l,\lambda,II}(x)$ , and by  $\nu_{II,1+it}^1(x)$  the contribution of the  $\mu_{l,\lambda}^1(x)$  to the sum defining  $\nu_{II,1+it}(x)$ .

Assuming for instance that  $D > 0$ , and keeping in mind that, according to (12.12),  $\lambda^{-1/3} D^{1/2} = Q_D(x, \delta)^{1/2}$  depends only on  $x$  and  $\delta$ , a Taylor expansion then shows that

$$\begin{aligned}
 |D + Ev|^{\frac{3}{2}} &= D^{\frac{3}{2}} + \frac{3}{2} D^{\frac{1}{2}} Ev + D^{-\frac{1}{2}} E^2 r_1(D^{-1} E, v), \\
 q_+ \left( \lambda^{-\frac{1}{3}} |D + Ev|^{\frac{1}{2}}, x, \delta \right) &= q_+ \left( \lambda^{-\frac{1}{3}} D^{\frac{1}{2}}, x, \delta \right) + \frac{1}{2} q'_+ \left( \lambda^{-\frac{1}{3}} D^{\frac{1}{2}}, x, \delta \right) \lambda^{-\frac{1}{3}} D^{-\frac{1}{2}} Ev \\
 &\quad + \lambda^{-\frac{1}{3}} D^{\frac{1}{2}} (D^{-1} E)^2 r_2 \left( \lambda^{-\frac{1}{3}} D^{\frac{1}{2}}, D^{-1} E, v, x, \delta \right),
 \end{aligned}$$

where  $q'_+$  denotes the partial derivative of  $q_+$  with respect to the first variable, and where  $r_1$  and  $r_2$  are smooth, real-valued functions. This implies that

$$(12.22) \quad |D + Ev|^{\frac{3}{2}} q_+ (\lambda^{-\frac{1}{3}} |D + Ev|^{\frac{1}{2}}, x, \delta) = D^{\frac{3}{2}} q_+ (\lambda^{-\frac{1}{3}} D^{\frac{1}{2}}, x, \delta) \\ + v \left[ \frac{1}{2} q'_+ (\lambda^{-\frac{1}{3}} D^{\frac{1}{2}}, x, \delta) \lambda^{-\frac{1}{3}} DE + \frac{3}{2} q'_+ (\lambda^{-\frac{1}{3}} D^{\frac{1}{2}}, x, \delta) D^{\frac{1}{2}} E \right] + r(\lambda^{-\frac{1}{3}} D^{\frac{1}{2}}, D^{-1}E, v, x, \delta),$$

again with a smooth, real-valued function  $r$ . In combination with (12.12) we then find that the complete phase in the oscillatory integral defining  $\mu_{l,\lambda}^1(x)$  is of the form

$$s_3[A' + B'v + r],$$

with  $A' = \lambda Q_{A'}(x, \delta)$ ,  $B' = 2^{\frac{l}{3}} \lambda^{\frac{2}{3}} Q_{B'}(x, \delta)$ , where

$$Q_{A'}(x, \delta) := Q_A(x, \delta) + Q_D(x, \delta)^{\frac{3}{2}} q_+ (Q_D(x, \delta)^{\frac{1}{2}}, x, \delta), \\ Q_{B'}(x, \delta) := Q_B(x, \delta) + \frac{1}{2} q'_+ (Q_D(x, \delta)^{\frac{1}{2}}, x, \delta) Q_D(x, \delta) \delta_0 \\ + 32 q'_+ (Q_D(x, \delta)^{\frac{1}{2}}, x, \delta) Q_D(x, \delta)^{\frac{1}{2}} \delta_0.$$

Thus, if  $|B'| \gg 1$ , then an integration by parts in  $v$  shows that

$$|\mu_{l,\lambda}^1(x)| \lesssim |D|^{-\frac{1}{4}} |B'|^{-1}.$$

This estimate allows to control the sum over all  $l$  such that  $|B'| \gg 1$ , and subsequently the sum over all dyadic  $\lambda$  such that  $|D| \gtrsim 1$ , and we arrive at the desired uniform estimate in  $x$  and  $\delta$ .

Next, if  $|B'| \lesssim 1$ , then we can argue in a similar way as before and apply Lemma 5.2 to the summation in  $l$  by putting here

$$H_{\lambda,x}(u_1, \dots, u_7) := \int e^{-is_3[A' + u_1 v + r(Q_D(x)^{\frac{1}{2}}, D^{-1}u_3, v, x, \delta)]} |D + u_2 v|^{-\frac{1}{4}} \chi_1(s_3) \chi_1(v) \\ \times a_+ (|Q_D(x, \delta) + u_3 v|^{\frac{1}{2}}, v, u_4, u_5, \delta) \chi_0(|u_6 + u_7 v|^{\frac{1}{2}}) dv ds_3$$

and choosing the cuboid  $Q$  in the obvious way. Then we easily see that  $\|H_{\lambda,x}\|_{C^1(Q)} \lesssim |D|^{-1/4}$ , and so after summation in those  $l$  for which  $|B'| \ll 1$ , we can also sum (absolutely) in the  $\lambda$ 's for which  $|D| \gtrsim 1$ . Observe that this requires again that  $\gamma(\zeta)$  contains the factor  $\gamma_1(\zeta)$ .

This concludes the proof of the uniform estimate of  $\nu_{II,1+it}^1(x)$  in  $x$  and  $\delta$ , and thus also of estimate (12.9).

**12.2.2. Contribution by the  $\nu_{l,I}^\lambda$ .** Let us next consider the contribution of the terms  $\nu_{l,I}^\lambda$  in (12.5), i.e., let us look at

$$(12.23) \quad \nu_{I,1+it} := \gamma(1+it) \sum_{2^M \leq \lambda \leq 2^M \rho^{-1}} \sum_{\{l: M_0 \leq 2^l \leq \frac{\lambda}{M_1}\}} 2^{-\frac{1}{2}itl} \lambda^{-2it} \mu_{l,\lambda,I}(x),$$

where we have set  $\mu_{l,\lambda,I} := 2^{-\frac{1}{3}l} \lambda^{-\frac{4}{3}} \nu_{l,I}^\lambda$ . We want to prove that



$$(12.24) \quad |\nu_{I,1+it}(x)| \leq C \quad \forall t \in \mathbb{R}, x \in \mathbb{R}^3.$$

Our discussion in Section 8 shows that

$$\begin{aligned} \mu_{l,\lambda,I}(x) &= \lambda^{\frac{1}{3}} 2^l \int e^{-is_3 \tilde{\Psi}(u,z,s_2,x,\delta,\lambda,l)} \tilde{a}(\sigma_{2^l \lambda^{-1}} u, (2^l \lambda^{-1})^{\frac{2}{3}} z, s_2, \delta) \chi_1(s_2, s_3) \\ &\times (1 - \chi_0) \left( \varepsilon (2^l \lambda^{-1})^{-\frac{1}{3}} (x_1 - s_2^{\frac{1}{n-2}} G_1(s_2, \delta)) \right) \chi_0(u) \chi_1(u_1) \chi_1(z) du_1 du_2 dz ds_2 ds_3, \end{aligned}$$

where  $\varepsilon > 0$  is small. Moreover,

$$\begin{aligned} \tilde{\Psi}(u, z, s_2, x, \delta, \lambda, l) &= \lambda^{\frac{1}{3}} 2^{\frac{2l}{3}} (x_1 - s_2^{\frac{1}{n-2}} G_1(s_2, \delta) - (2^l \lambda^{-1})^{\frac{1}{3}} u_1) z \\ &+ \lambda \left( s_2^{\frac{n}{n-2}} G_5(s_2, \delta) - x_1 s_2^{\frac{n-1}{n-2}} G_3(s_2, \delta) - s_2 x_2 - x_3 \right) \\ &+ 2^l (u_1^3 B_3(s_2, \delta_1, (2^{-l} \lambda)^{-\frac{1}{3}} u_1) + \phi_{2^{-l} \lambda}^\#(u_1, u_2, \tilde{\delta}^{2^{-l} \lambda}, s_2)), \end{aligned}$$

with  $\phi^\#$  given by (8.17). Observe that the first term is here much bigger than  $2^l$ , so that it dominates the third term.

We change variables from  $s_2$  to  $v := x_1 - s_2^{\frac{1}{n-2}} G_1(s_2, \delta)$ . Then  $s_2$  is a smooth function  $s_2(v, x_1, \delta)$ , and we may re-write

$$\begin{aligned} \mu_{l,\lambda,I}(x) &= \lambda^{\frac{1}{3}} 2^l \int e^{-is_3 \tilde{\Psi}_1(u,z,v,x,\delta,\lambda,l)} a_1(\sigma_{2^l \lambda^{-1}} u, (2^l \lambda^{-1})^{\frac{1}{3}} z, v, x, \delta) \chi_0(v) \chi_1(s_3) \\ &\times (1 - \chi_0) (\varepsilon (2^l \lambda^{-1})^{-\frac{1}{3}} v) \chi_0(u) \chi_1(u_1) \chi_1(z) du_1 du_2 dz v ds_3, \end{aligned}$$

where  $\tilde{\Psi}_1$  is of the form

$$\begin{aligned} \tilde{\Psi}_1(u, z, v, x, \delta, \lambda, l) &= (\lambda^{\frac{1}{3}} 2^{\frac{2l}{3}} v - 2^l u_1) z + 2^l g_1(u, v, x, (2^l \lambda^{-1})^{1/3}, \tilde{\delta}^{2^{-l} \lambda}, \delta) \\ &+ \lambda g_2(u, v, x, \delta), \end{aligned}$$

with smooth, real-valued functions  $g_1$  and  $g_2$ . Integrating  $N$  times by parts in  $z$ , and subsequently changing coordinates from  $v$  to  $w := (2^l \lambda^{-1})^{-\frac{1}{3}} v$ , we arrive at the following expression for  $\mu_{l,\lambda,I}(x)$ :

$$\begin{aligned} \mu_{l,\lambda,I}(x) &= 2^{(\frac{4}{3}-N)l} \int e^{-is_3 \Psi(y,z,w,x,\delta,\lambda,l)} a(\sigma_{2^l \lambda^{-1}} y, (2^l \lambda^{-1})^{\frac{1}{3}} z, (2^l \lambda^{-1})^{\frac{1}{3}} w, \delta) \chi_1(s_3) \\ (12.25) \quad &\times \chi_0((2^l \lambda^{-1})^{\frac{1}{3}} w) (1 - \chi_0)(\varepsilon w) \chi_0(y) \chi_1(y_1) \chi_1(z) \frac{1}{(w - y_1)^N} dy_1 dy_2 dz ds_3 dw, \end{aligned}$$

with phase  $\Psi$  of the form

$$\begin{aligned} \Psi(y, z, w, x, \delta, \lambda, l) &= 2^l (w - y_1) z + 2^l g_1(y, (2^l \lambda^{-1})^{\frac{1}{3}} w, x, (2^l \lambda^{-1})^{1/3}, \tilde{\delta}^{2^{-l} \lambda}, \delta) \\ (12.26) \quad &+ \lambda g_2(y, (2^l \lambda^{-1})^{\frac{1}{3}} w, x, \delta). \end{aligned}$$

Notice that we have also changed the names of variables  $u$  to  $y$ , in order to avoid possible confusion in the later application of Lemma 5.2. Recall also that in this integral,  $|w| \gg 1 \sim u_1$ .

A Taylor expansion of  $g_2$  in the “variable”  $(2^l \lambda^{-1})^{\frac{1}{3}} w$  then shows that we may re-write the phase in the form

$$(12.27) \quad \begin{aligned} \Psi(y, z, w, x, \delta, \lambda, l) &= 2^l \left[ (w - y_1)z + g_1(y, (2^l \lambda^{-1})^{\frac{1}{3}} w, x, (2^l \lambda^{-1})^{1/3}, \tilde{\delta}^{2^{-l}\lambda}, \delta) \right. \\ &\quad \left. h_0(y, (2^l \lambda^{-1})^{\frac{1}{3}} w, x, \delta) w^3 \right] \\ &\quad + \lambda h_3(y, x, \delta) + \lambda^{\frac{2}{3}} 2^{\frac{l}{3}} h_2(y, x, \delta) w + \lambda^{\frac{1}{3}} 2^{\frac{2l}{3}} h_1(y, x, \delta) w^2, \end{aligned}$$

where  $h_0, \dots, h_3$  are again smooth, real-valued functions of their (bounded) variables.

The following lemma will be useful.

**Lemma 12.2.** *Let  $\beta_1, \dots, \beta_n \in ]0, \infty[$  be given, pairwise distinct positive numbers. For any complex numbers  $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ , denote by  $\Lambda$  the set of all dyadic numbers  $\lambda = 2^j$  such that  $\max_{k=1, \dots, n} \lambda^{\beta_k} |\alpha_k| \geq 1$ . Then there exists an exceptional set  $\Lambda_e \subset \Lambda$  depending on the  $\alpha_k$  and  $\beta_k$  whose cardinality is bounded by a constant  $C_1(\beta_1, \dots, \beta_n)$  depending only on the  $\beta_1, \dots, \beta_n$  such that for all  $\lambda \in \Lambda \setminus \Lambda_e$  we have that  $|\sum_{k=1}^n \lambda^{\beta_k} \alpha_k| \geq 2/3$ , and moreover*

$$(12.28) \quad \sum_{\lambda \in \Lambda \setminus \Lambda_e} \left| \sum_{k=1}^n \lambda^{\beta_k} \alpha_k \right|^{-1} \leq C_2(\beta_1, \dots, \beta_n),$$

where the constant  $C_2(\beta_1, \dots, \beta_n)$  depends only on the numbers  $\beta_k$ .

*Proof.* We may assume without loss of generality that  $\alpha_k \neq 0$  for every  $k$ . Observe that if  $k \neq l$  and, say,  $\beta_k > \beta_l$ , then we have that  $1/4 \leq (2^{\beta_k j} |\alpha_k|) / (2^{\beta_l j} |\alpha_l|) \leq 4$  if and only if

$$\left| j + \frac{\log_2 |\frac{\alpha_k}{\alpha_l}|}{\beta_k - \beta_l} \right| \leq \frac{2}{\beta_k - \beta_l}.$$

We therefore define the set  $\Lambda_e$  to be the set of all dyadic numbers  $\lambda = 2^j \in \Lambda$  satisfying this conditions for at least on pair  $k \neq l$ . The cardinality of  $\Lambda_e$  is then clearly bounded by  $\binom{n}{2} 4 \max_{k \neq l} |\beta_k - \beta_l|^{-1}$ . Moreover, if  $\lambda \in \Lambda \setminus \Lambda_e$ , and if we choose a permutation  $(k(1), \dots, k(n))$  of  $(1, \dots, n)$  so that

$$\lambda^{\beta_{k(1)}} |\alpha_{k(1)}| > \lambda^{\beta_{k(2)}} |\alpha_{k(2)}| > \dots > \lambda^{\beta_{k(n)}} |\alpha_{k(n)}|,$$

then we have indeed even

$$\lambda^{\beta_{k(1)}} |\alpha_{k(1)}| > 4 \lambda^{\beta_{k(2)}} |\alpha_{k(2)}| > \dots > 4^{n-1} \lambda^{\beta_{k(n)}} |\alpha_{k(n)}|.$$

This implies that

$$\left| \sum_{k=1}^n \lambda^{\beta_k} \alpha_k \right| \geq \lambda^{\beta_{k(1)}} |\alpha_{k(1)}| \left( 1 - \sum_{l=1}^{n-1} 4^{-l} \right) \geq \frac{2}{3} \lambda^{\beta_{k(1)}} |\alpha_{k(1)}| \geq \frac{2}{3}.$$

And, since  $\sum_{j \in \mathbb{Z}: 2^{\beta_l j} |\alpha_l| \geq 1} (2^{\beta_l j} |\alpha_l|)^{-1} \leq (1 - 2^{-\beta_l})^{-1}$ , we obtain (12.17), with

$$C_2(\beta_1, \dots, \beta_n) := \frac{3}{2} n! \max_k \frac{1}{1 - 2^{-\beta_k}}.$$

Q.E.D.

In order to prove (12.24), let us consider

$$F(t, y, z, w, x, \delta, l) := \sum_{2^M \leq \lambda \leq 2^M \rho^{-1}} \lambda^{-2it} \int e^{-is_3 \Psi(y, z, w, x, \delta, \lambda, l)} \chi_1(s_3) ds_3$$

We shall prove that

$$(12.29) \quad |F(t, y, z, w, x, \delta, \lambda, l)| \leq C \frac{2^l (1 + |w|^3)}{|2^{-i2t} - 1|},$$

with a constant  $C$  not depending on  $t, y, z, x, \delta$  and  $l$ . By choosing  $N$  in (12.25) sufficiently big, we see that this estimate will imply (12.24), provided  $\gamma(\zeta)$  contains a factor

$$\gamma_4(\zeta) := \frac{2^{2(1-\zeta)} - 1}{3}.$$

Let us put  $\beta_3 := 1, \beta_2 := 2/3, \beta_1 := 1/3$  and, given  $y, w, x, \delta$  and  $l$ , also  $\alpha_3 := h_3(y, x, \delta), \alpha_2 := 2^{\frac{l}{3}} h_2(y, x, \delta)w, \alpha_1 := 2^{\frac{2l}{3}} h_1(y, x, \delta)w^2$ . Accordingly, we set

$$\begin{aligned} \Lambda_1 = \Lambda_1(y, w, x, \delta, l) &:= \left\{ \lambda = 2^j : 2^M \leq \lambda \leq 2^M \rho^{-1} \text{ and } \right. \\ &\quad \left. \max\{|\lambda h_3(y, x, \delta)|, \lambda^{\frac{2}{3}} |2^{\frac{l}{3}} h_2(y, x, \delta)w|, \lambda^{\frac{1}{3}} 2^{\frac{2l}{3}} |h_1(y, x, \delta)w^2|\} \geq 1 \right\} \\ &= \left\{ \lambda = 2^j : 2^M \leq \lambda \leq 2^M \rho^{-1} \text{ and } \max_{k=1, \dots, 3} \lambda^{\beta_k} |\alpha_k| \geq 1 \right\}, \end{aligned}$$

and

$$\begin{aligned} \Lambda_2 = \Lambda_2(y, w, x, \delta, l) &:= \left\{ \lambda = 2^j : 2^M \leq \lambda \leq 2^M \rho^{-1} \text{ and } \right. \\ &\quad \left. \max\{|\lambda h_3(y, x, \delta)|, \lambda^{\frac{2}{3}} |2^{\frac{l}{3}} h_2(y, x, \delta)w|, \lambda^{\frac{1}{3}} 2^{\frac{2l}{3}} |h_1(y, x, \delta)w^2|\} < 1 \right\}. \end{aligned}$$

We also denote by  $\Lambda_e \subset \Lambda$  the set of exceptional  $\lambda$ 's given by Lemma 12.2 for this choice of  $\beta_k$  and  $\alpha_k$ . Correspondingly, we decompose  $F = F_e + F_1 + F_2$ , where  $F_e, F_1$  and  $F_2$  are defined as  $F$ , only with summation over the dyadic  $\lambda$ 's restricted to the subsets  $\Lambda_e, \Lambda_1 \setminus \Lambda_e$  and  $\Lambda_2$ , respectively.

For  $F_e$ , we then trivially get the estimate  $|F(t, y, z, w, x, \delta, \lambda, l)| \leq C$ , since the cardinality of  $\Lambda_e$  is bounded by a constant not depending on the arguments of  $F_e$ .

Next, in order to estimate  $F_1$ , let us re-write

$$\int e^{-is_3 \Psi(y, z, w, x, \delta, \lambda, l)} \chi_1(s_3) ds_3 = \int e^{-is_3 (\lambda^{\beta_1} \alpha_1 + \lambda^{\beta_2} \alpha_2 + \lambda^{\beta_3} \alpha_3)} \left( e^{-is_3 \Psi_0(y, z, w, x, \delta, \lambda, l)} \chi_1(s_3) \right) ds_3,$$

where  $\Psi_0$  denotes the phase

$$\begin{aligned} \Psi_0(y, z, w, x, \delta, \lambda, l) := & 2^l \left[ (w - y_1)z + g_1(y, (2^l \lambda^{-1})^{\frac{1}{3}} w, x, (2^l \lambda^{-1})^{1/3}, \tilde{\delta}^{2^{-l}\lambda}, \delta) \right. \\ & \left. h_0(y, (2^l \lambda^{-1})^{\frac{1}{3}} w, x, \delta) w^3 \right]. \end{aligned}$$

Observe that  $|\Psi_0(y, z, w, x, \delta, \lambda, l)| \leq C 2^l (1 + |w|)$ . An integrating by parts in  $s_3$  therefore shows that

$$\left| \int e^{-is_3 \Psi(y, z, w, x, \delta, \lambda, l)} \chi_1(s_3) ds_3 \right| \leq C \frac{2^l (1 + |w|)}{|\lambda^{\beta_1} \alpha_1 + \lambda^{\beta_2} \alpha_2 + \lambda^{\beta_3} \alpha_3|}.$$

We may thus control the sum over all  $\lambda \in \Lambda_1$  by means of Lemma 12.2 and obtain the estimate

$$|F_1(t, y, z, w, x, \delta, \lambda, l)| \leq C 2^l (1 + |w|).$$

Finally,  $F_2$  can again be estimated by means of Lemma 5.2. Indeed, observe that in the sum defining  $F_2(t, y, z, w, x, \delta, \lambda, l)$ , the expressions

$$(2^l \lambda^{-1})^{\frac{1}{3}} w, (2^l \lambda^{-1})^{1/3}, \tilde{\delta}^{2^{-l}\lambda}, \lambda h_3(y, x, \delta), \lambda^{\frac{2}{3}} 2^{\frac{l}{3}} h_2(y, x, \delta) w, \lambda^{\frac{1}{3}} 2^{\frac{2l}{3}} h_1(y, x, \delta) w^2$$

are all uniformly bounded. Therefore, we may here put

$$H(u_1, \dots, u_6) := \int e^{-is_3 (2^l [(w - y_1)z + g_1(y, u_1, x, u_2, u_3, \delta) + h_0(y, u_1, x, \delta) w^3] + u_4 + u_5 + u_6)} \chi_1(s_3) ds_3,$$

with the  $a_k$  in Lemma 5.2 given by

$$a_1 := 2^{l/3} 2^w, a_2 := 2^{l/3}, \dots, a_4 := h_3(y, x, \delta), a_5 := 2^{\frac{l}{3}} h_2(y, x, \delta) w, a_6 := 2^{\frac{2l}{3}} h_1(y, x, \delta) w^2,$$

and the obvious corresponding cuboid  $Q$ . Then clearly  $\|H\|_{C^1(Q)} \leq C 2^l (1 + |w|^3)$ , and thus Lemma 5.2 yields the estimate

$$|F_2(t, y, z, w, x, \delta, \lambda, l)| \leq C \frac{2^l (1 + |w|^3)}{|2^{-i2t} - 1|}.$$

This concludes the proof of estimate (12.29), and thus also of (12.24).

**12.2.3. Contribution by the  $\nu_{l,III}^\lambda$ .** The contribution of the terms  $\nu_{l,III}^\lambda$  in (12.5) can be treated in a very similar way as the one by the terms  $\nu_{l,I}^\lambda$ . Indeed, arguing as before, we here arrive at the following expression for  $\mu_{l,\lambda,III} := 2^{-\frac{1}{3}l} \lambda^{-\frac{4}{3}} \nu_{l,III}^\lambda$ :

$$\begin{aligned} \mu_{l,\lambda,III}(x) &= 2^{(\frac{4}{3}-N)l} \int e^{-is_3 \Psi(y, z, w, x, \delta, \lambda, l)} a(\sigma_{2^l \lambda^{-1}} y, (2^l \lambda^{-1})^{\frac{1}{3}} z, (2^l \lambda^{-1})^{\frac{1}{3}} w, \delta) \chi_1(s_3) \\ (12.30) \quad &\times \chi_0((2^l \lambda^{-1})^{\frac{1}{3}} w) \chi_0\left(\frac{w}{\varepsilon}\right) \chi_0(y) \chi_1(y_1) \chi_1(z) \frac{1}{(w - y_1)^N} dy_1 dy_2 dz ds_3 dw. \end{aligned}$$

The phase  $\Psi$  is still given by (12.26). Notice that now  $|w| \ll 1 \sim u_1$ . The arguments used in the preceding subsection therefore carry over to this case, with minor modifications (even simplifications).

12.2.4. *Contribution by the  $\nu_{l,\infty}^\lambda$ .* Let us next look at

$$(12.31) \quad \nu_{\infty,1+it} := \gamma(1+it) \sum_{2^M \leq \lambda \leq 2^{M+\rho^{-1}}} \sum_{\{l: M_0 \leq 2^l \leq \frac{\lambda}{M_1}\}} 2^{-\frac{1}{2}itl} \lambda^{-2it} \mu_{l,\lambda,\infty}(x),$$

where we have set  $\mu_{l,\lambda,\infty} := 2^{-\frac{1}{3}l} \lambda^{-\frac{4}{3}} \nu_{l,\infty}^\lambda$ . We want to prove that

$$(12.32) \quad |\nu_{\infty,1+it}(x)| \leq C \quad \forall t \in \mathbb{R}, x \in \mathbb{R}^3.$$

To this end, recall formulas (8.39) and (8.40) for  $\nu_{l,\infty}^\lambda(x)$ . From these formulas, it is easy to see that  $|\nu_{l,\infty}^\lambda(x)| \lesssim 2^{-lN} \lambda^{-N}$  if  $|x| \gg 1$ , and thus summation in  $l$  and  $\lambda$  is no problem in this case. So, assume that  $|x| \lesssim 1$ . Then the second term in the phase  $\Psi(z, s_2, \delta)$  in (8.40) can be absorbed into the amplitude  $a_{N,l}$ , and we arrive at an expression of the following form for  $\mu_{l,\lambda,\infty}(x)$ :

$$\begin{aligned} \mu_{l,\lambda,\infty}(x) &= 2^{-lN} \lambda^{\frac{1}{3}} \int e^{-is_3 \lambda \left( s_2^{\frac{n}{n-2}} G_5(s_2, \delta) - x_1 s_2^{\frac{n-1}{n-2}} G_3(s_2, \delta) - s_2 x_2 - x_3 \right)} \\ &\quad \times a_{N,l} \left( z, s_2, s_3, \delta_0^r, \tilde{\delta}^{2^{-l}\lambda}, (2^{-l}\lambda)^{-\frac{1}{3}}, \lambda^{-\frac{1}{9}} \right) \chi_1(z) \chi_1(s_2) \chi_1(s_3) dz ds_2 ds_3, \end{aligned}$$

where  $a_{N,l}$  is a smooth function of all its (bounded) variables such that  $\|a_{N,l}\|_{C^k}$  is uniformly bounded in  $l$ . Denote by  $\Psi(s_2) = \Psi(s_2, x, \delta) = s_2^{\frac{n}{n-2}} G_5(s_2, \delta) - x_1 s_2^{\frac{n-1}{n-2}} G_3(s_2, \delta) - s_2 x_2 - x_3$  the phase appearing in this integral. We can now argue in a similar way as in Subsection 12.1:

Since  $s_2 \sim 1$  in the integral, we see that if  $|x_1| \ll 1$ , then  $|\partial_{s_2}^2 \Psi(s_2)| \sim 1$ , and van der Corput's estimate implies that  $|\mu_{l,\lambda,\infty}(x)| \lesssim 2^{-lN} \lambda^{1/3-1/2}$ . We can then sum the series (12.31) absolutely to arrive at (12.32). Let us therefore assume from now on that  $|x_1| \sim 1$ , and that the sign of  $x_1$  is such that there is a point  $s_2^c(x, \delta) \sim 1$  such that  $\partial_{s_2}^2 \Psi(s_2^c(x, \delta), x, \delta) = 0$ . This point is then unique, by the implicit function theorem, since  $|\partial_{s_2}^3 \Psi(s_2^c(x, \delta), x, \delta)| \sim 1$ . Changing coordinates from  $s_2$  to  $v := s_2 - s_2^c(x, \delta)$ , and applying a Taylor expansion in  $v$ , we see that the phase can be written in the form

$$Q_3(v, x, \delta) v^3 - Q_1(x, \delta) v + Q_0(x, \delta),$$

with smooth functions  $Q_j$ . Scaling in  $v$  by a factor  $\lambda^{-1/3}$  then leads to an expression of the following form for  $\mu_{l,\lambda,\infty}(x)$ :

$$\begin{aligned} \mu_{l,\lambda,\infty}(x) &= 2^{-lN} \int e^{-is_3 \left( Q_3(\lambda^{-\frac{1}{3}} w, x, \delta) w^3 + \lambda^{\frac{2}{3}} Q_1(x, \delta) w + \lambda Q_0(x, \delta) \right)} \\ (12.33) \quad &\times a_{N,l} \left( z, \lambda^{-\frac{1}{3}} w, s_3, \delta_0^r, \tilde{\delta}^{2^{-l}\lambda}, (2^{-l}\lambda)^{-\frac{1}{3}}, \lambda^{-\frac{1}{9}} \right) \chi_1(z) \chi_0(\lambda^{\frac{1}{3}} w) \chi_1(s_3) ds_3 dw dz. \end{aligned}$$

Performing first the integration in  $s_3$ , this easily implies the following estimate:

$$|\mu_{l,\lambda,\infty}(x)| \lesssim 2^{-lN} \int \int_{-\lambda^{\frac{1}{3}}}^{\lambda^{\frac{1}{3}}} \left( 1 + |Q_3(\lambda^{-\frac{1}{3}}w, x, \delta) w^3 + \lambda^{\frac{2}{3}}Q_1(x, \delta) w + \lambda Q_0(x, \delta)| \right)^{-N} \\ \times \chi_1(z) \chi_0(\lambda^{\frac{1}{3}}w) dw dz.$$

Putting  $A := \lambda Q_0(x, \delta)$ ,  $B := \lambda^{2/3}Q_1(x, \delta)$  and  $T := \lambda^{1/3}$  in Lemma 14.1, we then find that

$$|\mu_{l,\lambda,\infty}(x)| \lesssim 2^{-lN} \max\{|A|^{\frac{1}{3}}, |B|^{\frac{1}{2}}\}^{\epsilon - \frac{1}{2}}.$$

This estimate allows to sum over all  $\lambda$  such that  $\max\{|\lambda Q_0(x, \delta)|, \lambda^{2/3}|Q_1(x, \delta)|\} > 1$ , even absolutely.

There remains the summation over all  $\lambda$  such that  $|\lambda Q_0(x, \delta)| \leq 1$  and  $\lambda^{2/3}|Q_1(x, \delta)| \leq 1$ . However, in view of (12.33), this sum can easily be controlled by means of Lemma 5.2, as we have done in many similar cases before, and we shall therefore skip the details. Altogether, we arrive at (12.32).

**12.2.5. Contribution by the  $\nu_{l,00}^\lambda$ .** We finally come to the contribution of the terms  $\nu_{l,00}^\lambda$  in (12.5). Recall from Subsection 8.2.2 that

$$\widehat{\nu_{l,00}^\lambda}(\xi) := (2^{-l}\lambda)^{-\frac{2}{3}} \chi_1(s, s_3) \chi_1((2^{-l}\lambda)^{\frac{2}{3}} B_1(s, \delta_1)) e^{-is_3 \lambda B_0(s, \delta_1)} \\ \times \iint e^{-is_3 2^l \Phi(u_1, u_2, s, \delta, \lambda, l)} a(\sigma_{2^l \lambda^{-1}} u, \delta, s) \chi_0(u) \chi_0\left(\frac{u_1}{\varepsilon}\right) du_1 du_2,$$

where we assume  $\varepsilon > 0$  to be sufficiently small. Moreover, the phase  $\Phi$  is given by (8.34) and (8.35), with  $B = 3$ . Since

$$|\tilde{\delta}^{2^{-l}\lambda}| \ll 1 \quad \text{and} \quad (2^{-l}\lambda)^{\frac{2}{3}} |B_1(s, \delta_1)| \sim 1,$$

we see that we can again integrate by parts in  $u_1$ , in order to gain factors  $2^{-lN}$ , and then the same type of argument that led to the expression (8.36) for  $\widehat{\nu_{l,\infty}^\lambda}(\xi)$  can be applied in order to see that an analogous expression

$$\widehat{\nu_{l,00}^\lambda}(\xi) = 2^{-lN} \lambda^{-\frac{2}{3}} \chi_1(s, s_3) \chi_1((2^{-l}\lambda)^{\frac{2}{3}} B_1(s, \delta_1)) e^{-is_3 \lambda B_0(s, \delta_1)} \\ \times \tilde{a}_{N,l} \left( (2^{-l}\lambda)^{\frac{2}{3}} B_1(s, \delta_1), s, s_3, \tilde{\delta}^{2^{-l}\lambda}, \delta_0^r, (2^{-l}\lambda)^{-\frac{1}{3}}, \lambda^{-\frac{1}{3B}} \right),$$

can be obtained for  $\widehat{\nu_{l,00}^\lambda}(\xi)$  too, where  $\tilde{a}_{N,l}$  is again a smooth function of all its (bounded) variables such that  $\|a_{N,l}\|_{C^k}$  is uniformly bounded in  $l$ . From here on, we can argue exactly as for the  $\nu_{l,\infty}^\lambda$ .

This concludes the proof of the second estimate in (11.18), and thus also the proof of part (a) of Proposition 11.3.

### 13. PROOF OF PROPOSITION 11.3 (B) : COMPLEX INTERPOLATION

In this section, we assume that  $B = 3$  and  $A = 0$  in (6.20), and that  $\lambda\rho \gg 1$ .

13.1. **Estimation of  $T_{\delta, Ai}^{III}$ .** As usually, we embed  $\nu_{\delta, Ai}^{III}$  into an analytic family of measures

$$\nu_{\delta, \zeta}^{III} := \gamma(\zeta) \sum_{2^M \rho^{-1} < \lambda \leq 2^{-M} \delta_0^{-3}} (\rho^{-\frac{4}{5}}(\lambda \rho))^{\frac{5}{6}(1-3\zeta)} \nu_{\delta, 0}^{\lambda},$$

where  $\zeta$  lies in the complex strip  $\Sigma$  given by  $0 \leq \operatorname{Re} \zeta \leq 1$ . Since the supports of the  $\widehat{\nu_{\delta, 0}^{\lambda}}$  are almost disjoint, and since, according to (11.4),  $\|\nu_{\delta, 0}^{\lambda}\|_{\infty} \lesssim \rho^{-4/3}(\lambda \rho)^{5/3}$ , we see that

$$\|\widehat{\nu_{\delta, it}^{III}}\|_{\infty} \lesssim 1 \quad \forall t \in \mathbb{R}.$$

Again, by Stein's interpolation theorem, it will therefore suffice to prove the following estimate:

$$(13.1) \quad |\nu_{\delta, 1+it}^{III}(x)| \leq C \quad \forall t \in \mathbb{R}, x \in \mathbb{R}^3.$$

Now, if  $|x| \gg 1$ , arguing in a similar way as for the case  $B = 4$  in Subsection 10.1, we see that  $|\nu_{\delta, 0}^{\lambda}(x)| \lesssim \rho^{\frac{2}{3} + \frac{2}{3}} \lambda^3 (\lambda \rho^{2/3})^{-N}$  for every  $N \in \mathbb{N}$ , which allows to sum absolutely in  $\lambda$  and obtain (13.1).

From now on, we shall therefore assume that  $|x| \lesssim 1$ . We then re-write

$$(13.2) \quad \nu_{\delta, 1+it}^{III}(x) = \gamma(1+it) \sum_{2^M \rho^{-1} < \lambda \leq 2^{-M} \delta_0^{-3}} (\rho^{-\frac{4}{5}}(\lambda \rho))^{-\frac{5}{2}it} \mu_{\lambda}(x),$$

where  $\mu_{\lambda} := \rho^{4/3}(\lambda \rho)^{-5/3} \nu_{\delta, 0}^{\lambda}$ . Recall also from (11.7) that  $\|\nu_{\delta, 0}^{\lambda}\|_{\infty} \lesssim \rho^{-4/3}(\lambda \rho)^{5/3}$ , which barely fails to be sufficient to obtain (13.1).

We therefore need again a more refined reasoning. Following our discussion for the case  $B = 4$  in Subsection 10.1, we decompose  $\nu_{\delta, 0}^{\lambda} = \nu_{0, I}^{\lambda} + \nu_{0, II}^{\lambda}$  as in (10.16). In the same way in which we had derived (10.32), we find here that

$$(13.3) \quad \|\nu_{0, I}^{\lambda}(x)\|_{\infty} \leq C_N \rho^{\frac{2}{3}} \lambda^2 (\lambda \rho)^{-N}.$$

If we denote by  $\mu_{\lambda, I} := \rho^{4/3}(\lambda \rho)^{-5/3} \nu_{0, I}^{\lambda}$ , then this estimate shows that we can sum the corresponding series in (13.2), with  $\mu_{\lambda}$  replaced by  $\mu_{\lambda, I}$ , absolutely, and obtain the desired uniform estimate in  $x$  and  $\delta$ .

What remains are the contributions by the  $\nu_{0, II}^{\lambda}$ . In order to keep the notation simple, we therefore shall assume from now on that  $\mu_{\lambda} = \rho^{4/3}(\lambda \rho)^{-5/3} \nu_{0, II}^{\lambda}$ .

In analogy with our formulas (10.22) and (10.23), we then find the following expressions for  $\mu_{\lambda}$ :

$$\begin{aligned} \mu_{\lambda}(x) &= (\lambda \rho)^{\frac{1}{3}} \int e^{-i\lambda s_3 \Phi_4(y_1, u_2, w, x, \delta)} \widehat{\chi_0}(s_3 y_1) \chi_0(w) \chi_0(w - (\lambda \rho)^{-1} y_1) \\ &\quad \times \chi_0(u_2) a_4((\lambda \rho^{\frac{2}{3}})^{-1} y_1, \rho^{\frac{1}{3}} u_2, w, s_1, \rho^{\frac{1}{3}}, x, \delta) \tilde{\chi}_1(s_3) dy_1 du_2 dw ds_3 \end{aligned}$$

with phase  $\Phi_4$  of the form

$$\begin{aligned} \Phi_4(y_1, u_2, w, x, \delta) = & \Psi_3(\rho^{\frac{1}{3}}w, x, \delta) + \rho(w - (\lambda\rho)^{-1}y_1)^3 \tilde{B}_3(\rho^{\frac{1}{3}}w, (\lambda\rho^{\frac{2}{3}})^{-1}y_1, x, \delta) \\ & + \rho\left(u_2^3 b(\rho^{\frac{1}{3}}w, (\lambda\rho^{\frac{2}{3}})^{-1}y_1, \rho^{\frac{1}{3}}u_2, x, \delta_0^{\mathfrak{r}}) + \delta'_{3,0}u_2 \tilde{\alpha}_1(\rho^{\frac{1}{3}}w, x, \delta_0^{\mathfrak{r}}) \right. \\ & \left. + \delta'_0 u_2 (w - (\lambda\rho)^{-1}y_1) \alpha_{1,1}(\rho^{\frac{1}{3}}w, (\lambda\rho^{\frac{2}{3}})^{-1}y_1, x, \delta_0^{\mathfrak{r}})\right). \end{aligned}$$

Recall that in this integral,  $|u_2| + |w| \lesssim 1$  and  $|y_1| \lesssim \lambda\rho$ . Moreover, the factor  $\widehat{\chi}_0(s_3 y_1)$  guarantees the absolute convergence of this integral with respect to the variable  $y_1$ . We also recall that  $\delta'_0 + \delta'_{3,0} \sim 1$ , and that the coefficient  $\delta'_{3,0}$  does not appear in Case ND, where  $\alpha_{1,1} = 0$ . Finally, we perform the change of variables  $u_2 = (\lambda\rho)^{-1/3}y_2$  and obtain

$$\begin{aligned} \mu_\lambda(x) = & \int e^{-is_3 \Phi_5(y, w, x, \delta; \lambda)} \widehat{\chi}_0(s_3 y_1) \chi_0(w) \chi_0(w - (\lambda\rho)^{-1}y_1) \\ (13.4) \quad & \times \chi_0((\lambda\rho)^{-1/3}y_2) a_5((\lambda\rho^{\frac{2}{3}})^{-1}y_1, \lambda^{-\frac{1}{3}}y_2, w, s_1, \rho^{\frac{1}{3}}, x, \delta) \tilde{\chi}_1(s_3) ds_3 dy_1 dy_2 dw, \end{aligned}$$

with phase  $\Phi_5$  of the form

$$\begin{aligned} \Phi_5(y, w, x, \delta; \lambda) = & \lambda \Psi_3(\rho^{\frac{1}{3}}w, x, \delta) + \lambda \rho(w - (\lambda\rho)^{-1}y_1)^3 \tilde{B}_3(\rho^{\frac{1}{3}}w, (\lambda\rho^{\frac{2}{3}})^{-1}y_1, x, \delta) \\ (13.5) \quad & + y_2^3 \tilde{b}(\rho^{\frac{1}{3}}w, (\lambda\rho^{\frac{2}{3}})^{-1}y_1, \lambda^{-\frac{1}{3}}y_2, x, \delta_0^{\mathfrak{r}}) \\ & + y_2 (\lambda\rho)^{\frac{2}{3}} \left( \delta'_{3,0} \tilde{\alpha}_1(\rho^{\frac{1}{3}}w, x, \delta_0^{\mathfrak{r}}) + \delta'_0 (w - (\lambda\rho)^{-1}y_1) \tilde{\alpha}_{1,1}(\rho^{\frac{1}{3}}w, (\lambda\rho^{\frac{2}{3}})^{-1}y_1, x, \delta_0^{\mathfrak{r}}) \right). \end{aligned}$$

Performing a Taylor expansion with respect to the bounded quantities  $(\lambda\rho^{\frac{2}{3}})^{-1}y_1$  and  $(\lambda\rho)^{-1}y_1$ , we see that we may re-write the phase as

$$\begin{aligned} \Phi_5(y, w, x, \delta; \lambda) = & A + B y_2 + \tilde{b}(\rho^{\frac{1}{3}}w, (\lambda\rho^{\frac{2}{3}})^{-1}y_1, \lambda^{-\frac{1}{3}}y_2, x, \delta_0^{\mathfrak{r}}) y_2^3 \\ (13.6) \quad & + r_1(y_1) + (\lambda\rho)^{-\frac{1}{3}} y_2 r_2(y_1), \end{aligned}$$

where

$$\begin{aligned} A := & \lambda \left[ \Psi_3(\rho^{\frac{1}{3}}w, x, \delta) + \rho w^3 \tilde{B}_3(\rho^{\frac{1}{3}}w, 0, x, \delta) \right] =: \lambda Q_A(\rho^{\frac{1}{3}}w, x, \delta); \\ B := & (\lambda\rho)^{\frac{2}{3}} \left( \delta'_{3,0} \tilde{\alpha}_1(\rho^{\frac{1}{3}}w, x, \delta_0^{\mathfrak{r}}) + \delta'_0 w \tilde{\alpha}_{1,1}(\rho^{\frac{1}{3}}w, 0, x, \delta_0^{\mathfrak{r}}) \right) =: \lambda^{\frac{2}{3}} Q_B(\rho^{\frac{1}{3}}w, x, \delta), \end{aligned}$$

and where  $r_1$  and  $r_2$  are of the form

$$\begin{aligned} r_1(y_1) = & R_1(w, (\lambda\rho)^{-1}y_1, \rho^{\frac{1}{3}}, x, \delta) y_1, \\ r_2(y_1) = & R_2(w, (\lambda\rho)^{-1}y_1, \rho^{\frac{1}{3}}, x, \delta) y_1, \end{aligned}$$

with smooth functions  $R_1, R_2$  of their bounded entries  $w, (\lambda\rho)^{-1}y_1, \rho^{\frac{1}{3}}, x$  and  $\delta$ .

Let us put, for  $w$  fixed such that  $|w| \lesssim 1$ ,

$$\begin{aligned} \mu_\lambda(w, x) := & \int e^{-is_3 \Phi_5(y, w, x, \delta; \lambda)} \widehat{\chi}_0(s_3 y_1) \chi_0(w - (\lambda\rho)^{-1}y_1) \\ & \times \chi_0((\lambda\rho)^{-1/3}y_2) a_5((\lambda\rho^{\frac{2}{3}})^{-1}y_1, \lambda^{-\frac{1}{3}}y_2, w, s_1, \rho^{\frac{1}{3}}, x, \delta) \tilde{\chi}_1(s_3) ds_3 dy_1 dy_2. \end{aligned}$$



By means of integrations by parts in  $s_3$  and exploiting the rapid decay of  $\widehat{\chi_0}(s_3 y_1)$  in  $y_1$ , we may estimate

$$|\mu_\lambda(w, x)| \leq C_N \int_{|y_2| \lesssim (\lambda \rho)^{\frac{1}{3}}} \int \int \left( 1 + |A + B y_2 + \tilde{b}(\rho^{\frac{1}{3}} w, (\lambda \rho^{\frac{2}{3}})^{-1} y_1, \lambda^{-\frac{1}{3}} y_2, x, \delta_0^x)| y_2^3 + r_1(y_1) + (\lambda \rho)^{-\frac{1}{3}} y_2 r_2(y_1) \right)^{-N} (1 + |y_1|)^{-N} dy_1 dy_2.$$

Observe first that  $|B| \lesssim (\lambda \rho)^{\frac{2}{3}}$ . Thus, if  $|A| \gg \lambda \rho$ , then the term  $A$  becomes dominant, and we can clearly estimate  $|\mu_\lambda(x)| \leq C|A|^{-N}$  for every  $N \in \mathbb{N}$ . Otherwise, if  $|A| \lesssim \lambda \rho$ , then by choosing  $T := c(\lambda \rho)^{1/3}$  in Lemma 14.1, with a suitable constant  $c > 0$ , we see that all assumptions of this lemma are satisfied, and we obtain the estimate

$$(13.7) \quad |\mu_\lambda(w, x)| \leq C \max\{|A|^{\frac{1}{3}}, |B|^{\frac{1}{2}}\}^{-\frac{1}{2}}.$$

This estimate thus holds no matter how large  $|A|$  is.

Consider the function

$$F(t, w, x, \delta) := \sum_{2^M \rho^{-1} < \lambda \leq 2^{-M} \delta_0^{-3}} (\rho^{-\frac{4}{5}}(\lambda \rho))^{-\frac{5}{2}it} \mu_\lambda(w, x),$$

for  $|w| \lesssim 1$ . We shall prove that

$$(13.8) \quad |F(t, w, x, \delta)| \leq C \frac{1}{|2^{-i\frac{5}{2}t} - 1|},$$

with a constant  $C$  not depending on  $t, w, x$  and  $\delta$ . This estimate will immediately yield the desired estimate for the contributions of the  $\nu_{0,II}^\lambda$  and thus complete the proof of (13.1), provided we choose

$$\gamma(\zeta) := \frac{2^{\frac{5}{2}(1-\zeta)} - 1}{2^{\frac{5}{3}} - 1}.$$

Given  $w, x, \delta$ , denote by  $\Lambda(w, x, \delta)$  the set of all dyadic  $\lambda$  from our summation range  $\Lambda := \{\lambda = 2^j : 2^M \rho^{-1} < \lambda \leq 2^{-M} \delta_0^{-3}\}$ , for which either  $|A| = \lambda |Q_A(\rho^{\frac{1}{3}} w, x, \delta)| > 1$  or  $|B| = \lambda^{\frac{2}{3}} |Q_B(\rho^{\frac{1}{3}} w, x, \delta)| > 1$ . We then decompose  $F(t, w, x, \delta) = F_1(t, w, x, \delta) + F_2(t, w, x, \delta)$ , where  $F_1(t, w, x, \delta)$  and  $F_2(t, w, x, \delta)$  are defined as  $F$ , only with summation restricted to the subsets  $\Lambda(w, x, \delta)$  and  $\Lambda \setminus \Lambda(w, x, \delta)$ , respectively. Then, by (13.7), we clearly have that

$$|F_1(t, w, x, \delta)| \leq \sum_{\lambda \in \Lambda(w, x, \delta)} |\mu_\lambda(w, x)| \leq C,$$

and we are thus left with  $F_2(t, w, x, \delta)$ .

In the corresponding sum, we have  $\lambda |Q_A(\rho^{\frac{1}{3}} w, x, \delta)| \leq 1$  and  $\lambda^{\frac{2}{3}} |Q_B(\rho^{\frac{1}{3}} w, x, \delta)| \leq 1$ , and therefore  $F_2$  can again be estimated by means of Lemma 5.2. Indeed, we may here

put

$$\begin{aligned} H(u_1, \dots, u_6) \\ := \int e^{-is_3 \left( u_1 + u_2 y_2 + \tilde{b}(\rho^{\frac{1}{3}} w, u_3 y_1, u_4 y_2, x, \delta_0^x) y_2^3 + R_1(w, u_5 y_1, \rho^{\frac{1}{3}} x, \delta) y_1 + u_6 y_2 R_2(w, u_5 y_1, \rho^{\frac{1}{3}} x, \delta) \right)} \\ \widehat{\chi_0}(s_3 y_1) \chi_0(w - u_5 y_1) \chi_0(u_6 y_2) a_5(u_3 y_1, u_4 y_2, w, s_1, \rho^{\frac{1}{3}} x, \delta) \tilde{\chi}_1(s_3) ds_3 dy_1 dy_2, \end{aligned}$$

where the variables  $u_1, \dots, u_6$  correspond to the bounded expressions  $\lambda Q_A(\rho^{\frac{1}{3}} w, x, \delta)$ ,  $\lambda^{\frac{2}{3}} Q_B(\rho^{\frac{1}{3}} w, x, \delta)$ ,  $(\lambda \rho^{2/3})^{-1}$ ,  $\lambda^{-1/3}$ ,  $(\lambda \rho)^{-1}$  and  $(\lambda \rho)^{-1/3}$ , respectively. By means of integrations by parts in the variable  $y_2$  for  $|y_2| \gg 1$  (or, alternatively, in  $s_3$ ), it is then easily verified that  $\|H\|_{C^1(Q)} \leq C$ , where  $Q$  denotes the obvious cuboid  $Q$  appearing in this situation. Thus, estimate (13.8) follows from Lemma 5.2.

**13.2. Estimation of  $T_\delta^{IV}$ .** The estimation of the operator  $T_\delta^{IV}$  will follow similar ideas as the one for  $T_\delta^{II}$ . Nevertheless, for the convenience of the reader, we will give some details.

As usually, we embed  $\nu_\delta^{IV}$  into an analytic family of measures

$$\nu_{\delta, \zeta}^{IV} := \gamma(\zeta) \sum_{\{l: M_0 \leq 2^l \leq \frac{\rho^{-1}}{M_1}\}} \sum_{2^M \rho^{-1} < \lambda \leq 2^{-M} \delta_0^{-3}} \left( (2^l \rho)^{-\frac{4}{5}} (\lambda 2^l \rho) \right)^{\frac{5}{6}(1-3\zeta)} \nu_{l,0}^\lambda,$$

where  $\zeta$  lies in the complex strip  $\Sigma$  given by  $0 \leq \operatorname{Re} \zeta \leq 1$ . Since the supports of the  $\widehat{\nu_{\delta,0}^\lambda}$  are almost disjoint, estimate (11.12) shows that

$$\|\widehat{\nu_{\delta, it}^{IV}}\|_\infty \lesssim 1 \quad \forall t \in \mathbb{R}.$$

Again, by Stein's interpolation theorem, it will therefore suffice to prove the following estimate:

$$(13.9) \quad |\nu_{\delta, 1+it}^{IV}(x)| \leq C \quad \forall t \in \mathbb{R}, x \in \mathbb{R}^3,$$

where we write

$$(13.10) \quad \nu_{\delta, 1+it}^{IV}(x) = \gamma(1+it) \sum_{\{l: M_0 \leq 2^l \leq \frac{\rho^{-1}}{M_1}\}} \sum_{2^M \rho^{-1} < \lambda \leq 2^{-M} \delta_0^{-3}} (\lambda (2^l \rho)^{\frac{1}{5}})^{-\frac{5}{2}it} \mu_{l,\lambda}(x),$$

with  $\mu_{l,\lambda} := \lambda^{-5/3} (2^l \rho)^{-1/3} \nu_{l,0}^\lambda$ .

Regretfully, it seems that the approach in the previous subsection cannot be applied in the present situation, and a more refined analysis is needed, similar to our discussion in Subsection 12.2. From (11.13) to (11.15) we get that

$$\mu_{l,\lambda} = (2^l \rho) \lambda^{\frac{4}{3}} \int e^{-i\lambda s_3 \Phi_2(u, z, s_2, x, \delta)} \chi_1(z) \chi_0(u) a(\sigma_{2^l \rho} u, (2^l \rho)^{\frac{2}{3}} z, s, \delta) \tilde{\chi}_1(s_2, s_3) du dz ds_2 ds_3,$$

where

$$\begin{aligned}
 \Phi_2(u, z, s_2, x, \delta) &= s_2^{\frac{n}{n-2}} G_5(s_2, \delta) - x_1 s_2^{\frac{n-1}{n-2}} G_3(s_2, \delta) - s_2 x_2 - x_3 \\
 (13.11) \quad &+ z(2^l \rho) \left( (2^l \rho)^{-\frac{1}{3}} (x_1 - s_2^{\frac{1}{n-2}} G_1(s_2, \delta)) - u_1 \right) + (2^l \rho) u_1^3 B_3(s_2, \delta_1, (2^l \rho)^{\frac{1}{3}} u_1) \\
 &+ (2^l \rho) \left( u_2^3 b(\sigma_{2^l \rho} u, \delta_0^r, s_2) + \delta'_{3,0} u_2 \tilde{\alpha}_1(\delta_0^r, s_2) + \delta'_0 u_1 u_2 \alpha_{1,1}((2^l \rho)^{\frac{1}{3}} u_1, \delta_0^r, s_2) \right).
 \end{aligned}$$

Now, if  $|x| \gg 1$ , we see that  $|\mu_{l,\lambda}(x)| \lesssim 2^l \rho \lambda^{4/3} (\lambda(2^l \rho)^{2/3})^{-N}$  for every  $N \in \mathbb{N}$ , which is stronger than what is needed for (13.9).

From now on we shall therefore assume that  $|x| \lesssim 1$ . For such  $x$  fixed, we again decompose

$$(13.12) \quad \nu_{l,0}^\lambda = \nu_{l,I}^\lambda + \nu_{l,II}^\lambda,$$

where  $\nu_{l,I}^\lambda$  and  $\nu_{l,II}^\lambda$  denote the contributions to the integral above by the region  $L_I$  where  $|x_1 - s_2^{\frac{1}{n-2}} G_1(s_2, \delta)| \gg (2^l \rho)^{\frac{1}{3}}$ , and the region  $L_{II}$  where  $|x_1 - s_2^{\frac{1}{n-2}} G_1(s_2, \delta)| \lesssim (2^l \rho)^{\frac{1}{3}}$ , respectively. Then, in analogy with (13.3), by means of integrations by parts in  $z$  we obtain

$$(13.13) \quad \|\nu_{l,I}^\lambda(x)\|_\infty \leq C_N = (2^l \rho)^{\frac{2}{3}} \lambda^2 (\lambda 2^l \rho)^{-N}.$$

If we denote by  $\mu_{l,\lambda,I} := \lambda^{-5/3} (2^l \rho)^{-1/3} \nu_{l,I}^\lambda$ , then this estimate shows that we can sum the corresponding series in (13.10), with  $\mu_{l,\lambda}$  replaced by  $\mu_{l,\lambda,I}$ , absolutely, and obtain the desired uniform estimate in  $x$  and  $\delta$ .

What remains are the contributions by the  $\nu_{l,II}^\lambda$ . In order to keep the notation simple, we therefore shall assume from now on that  $\mu_{l,\lambda} = \lambda^{-5/3} (2^l \rho)^{-1/3} \nu_{l,I}^\lambda$ , i.e., that

$$\begin{aligned}
 \mu_{l,\lambda}(x) &= (2^l \rho) \lambda^{\frac{4}{3}} \int e^{-i\lambda s_3 \Phi_2(u, z, s_2, x, \delta)} a((2^l \rho)^{\frac{1}{3}} u, (2^l \rho)^{\frac{2}{3}} z, s, \delta) \chi_1(z) \chi_0(u) \\
 (13.14) \quad &\chi_0((2^l \rho)^{-\frac{1}{3}} (x_1 - s_2^{\frac{1}{n-2}} G_1(s_2, \delta))) \tilde{\chi}_1(s_2, s_3) du dz ds_2 ds_3,
 \end{aligned}$$

Given a point  $u^0, s_2^0, z^0$  such that the amplitude in this integral does not vanish, we want to understand the contribution of a small neighborhood of this point to the integral. Assume first that  $\partial_{u_1} \Phi_2(u^0, z^0, s_2^0, x, \delta) \neq 0$ . Then, integrations by parts in  $u_1$  allow to gain factors  $(\lambda 2^l \rho)^{-N}$ , and so we can again sum the corresponding contributions to  $\nu_{\delta, 1+it}^{IV}(x)$  absolutely.

Let us next assume that  $\partial_{u_1} \Phi_2(u^0, z^0, s_2^0, x, \delta) = 0$ . For a short while, it will then be helpful to change coordinates from  $s_2$  first to  $v := x_1 - s_2^{\frac{1}{n-2}} G_1(s_2, \delta)$ , and then to  $w := (2^l \rho)^{-\frac{1}{3}} v = (2^l \rho)^{-\frac{1}{3}} (x_1 - s_2^{\frac{1}{n-2}} G_1(s_2, \delta))$  in a similar way is in Subsection 10.1, and re-write

$$\begin{aligned}
 \mu_{l,\lambda}(x) &= (\lambda 2^l \rho)^{\frac{4}{3}} \int e^{-i\lambda s_3 \tilde{\Phi}_2(u, z, w, s_2, x, \delta)} a((2^l \rho)^{\frac{1}{3}} u, (2^l \rho)^{\frac{2}{3}} z, (2^l \rho)^{\frac{1}{3}} w, x, \delta) \chi_1(z) \chi_0(u) \\
 &\chi_0(w)) \chi_1(s_3) du dz ds_2 ds_3,
 \end{aligned}$$

where

$$\begin{aligned}\tilde{\Phi}_2 = & z(2^l \rho)(w - u_1) + \Psi_3(\rho^{\frac{1}{3}} w, x, \delta) + (2^l \rho) u_1^3 B_3((2^l \rho)^{\frac{1}{3}} w, (2^l \rho)^{\frac{1}{3}} u_1, x, \delta) \\ & + (2^l \rho) \left( u_2^3 b((2^l \rho)^{\frac{1}{3}} u, (2^l \rho)^{\frac{1}{3}} w, x, \delta_0^{\mathfrak{r}}) + \delta'_{3,0} u_2 \tilde{\alpha}_1((2^l \rho)^{\frac{1}{3}} w, x, \delta_0^{\mathfrak{r}}) \right. \\ & \left. + \delta'_0 u_1 u_2 \alpha_{1,1}((2^l \rho)^{\frac{1}{3}} u_1, (2^l \rho)^{\frac{1}{3}} w, x, \delta_0^{\mathfrak{r}}) \right),\end{aligned}$$

By  $w^0$  we denote the value of  $w$  corresponding to  $s_2^0$ . We now can see that there is also a unique critical point of the phase with respect to the variable  $z$ , at  $z^0$ , provided  $w = u_1^0$ . Thus, if  $w^0 = u_1^0$ , then the phase has a critical point with respect to the variable  $(u_1, s_2)$  respectively  $(u_1, w)$ ; otherwise, we can again integrate by parts in  $z$ , which allows to gain factors  $(\lambda 2^l \rho)^{-N}$  as before, and we are done. So, assume that  $w^0 = u_1^0$ . Since the phase is linear in  $z$ , and since  $|\partial_z \partial_{u_1} \tilde{\Phi}_2| \sim 2^l \rho$  at the critical point, we see that we may apply the method of stationary phase to the double integration with respect to the variables  $(u_1, z)$  and gain in particular a factor  $(\lambda 2^l \rho)^{-1}$ . Having realized this, we may come back to our previous formula (13.14), and knowing that we may apply the method of stationary phase to the integration with respect to the variables  $(u_1, z)$  as well, we see that we may essentially write

$$\begin{aligned}\mu_{l,\lambda}(x) = & \lambda^{\frac{1}{3}} \int e^{-i\lambda s_3 \Psi_2(u_2, s_2, x, \delta, l)} a_2((2^l \rho)^{\frac{1}{3}} u_2, (2^l \rho)^{\frac{1}{3}} s, \delta) \chi_0(u_2) \\ (13.15) \quad & \chi_0((2^l \rho)^{-\frac{1}{3}}(x_1 - s_2^{\frac{1}{n-2}} G_1(s_2, \delta))) \tilde{\chi}_1(s_2, s_3) du_2 ds_2 ds_3,\end{aligned}$$

where the phase  $\Psi_2$  arises from  $\Phi_2$  by replacing  $(u_1, z)$  by the critical point  $(u_1^0, z^0)$ .

Now, arguing exactly as in Subsection 10.1, by means of Lemma 7.1 in [21] we find that the phase  $\Psi_2$  is given by the expression in (8.52), with  $B = 3$ , i.e.,

$$\Psi_2 = s_2 x_1^2 \omega(\delta_1 x_1) + x_1^n \alpha(\delta_1 x_1) + s_2 \delta_0 y_2 + y_2^3 b(x_1, y_2, \delta) + r(x_1, y_2, \delta) - s_2 x_2 - x_3.$$

Moreover, since we here have changed coordinates from  $y_2$  to  $u_2$  so that  $y_2 = (2^l \rho)^{1/3} u_2$ , this means that

$$\begin{aligned}\Psi_2(u_2, s_2, x, \delta, l) = & s_2 x_1^2 \omega(\delta_1 x_1) + x_1^n \alpha(\delta_1 x_1) - s_2 x_2 - x_3 \\ (13.16) \quad & + (2^l \rho) u_2^3 b(x_1, (2^l \rho)^{1/3} u_2, \delta) + (2^l \rho)^{1/3} u_2 [\delta_0 s_2 + \delta_3 x_1^{n_1} \alpha_1(\delta_1 x_1)]\end{aligned}$$

(compare (8.4)). Note that  $\partial_{s_2}(s_2^{\frac{1}{n-2}} G_1(s_2, \delta)) \sim 1$  because  $s_2 \sim 1$  and  $G_1(s_2, 0) = 1$ . Therefore, the relation  $|x_1 - s_2^{\frac{1}{n-2}} G_1(s_2, \delta)| \lesssim (2^l \rho)^{1/3}$  can be re-written as  $|s_2 - \tilde{G}_1(x_1, \delta)| \lesssim (2^l \rho)^{1/3}$ , where  $\tilde{G}_1$  is again a smooth function such that  $|\tilde{G}_1| \sim 1$ . If we write

$$s_2 = (2^l \rho)^{\frac{1}{3}} v + \tilde{G}_1(x_1, \delta),$$

then this means that  $|v| \lesssim 1$ . We shall therefore change variables from  $s_2$  to  $v$ , which leads to the following expression for  $\mu_{l,\lambda}(x)$  :

$$\begin{aligned} \mu_{l,\lambda}(x) &= (\lambda 2^l \rho)^{\frac{1}{3}} \int e^{-i\lambda s_3 \Psi_3(u_2, v, x, \delta, l)} a_3((2^l \rho)^{\frac{1}{3}} u_2, (2^l \rho)^{\frac{1}{3}} v, x, \delta) \\ &\quad \times \chi_0(u_2) \chi_0(v) \chi_1(s_3) du_2 dv ds_3, \end{aligned}$$

with a smooth amplitude  $a_3$  and the new phase function

$$\begin{aligned} \Psi_3(u_2, v, x, \delta, l) &= v (2^l \rho)^{\frac{1}{3}} (x_1^2 \omega(\delta_1 x_1) - x_2) + (2^l \rho)^{\frac{2}{3}} \delta_0 v u_2 + Q_A(x, \delta) \\ &\quad + (2^l \rho) u_2^3 b(x_1, (2^l \rho)^{\frac{1}{3}} u_2, \delta) + (2^l \rho)^{\frac{1}{3}} u_2 Q_D(x, \delta) \end{aligned}$$

(compare with the corresponding expressions in (12.10) to (12.12)). Finally, putting  $y_2 := (\lambda 2^l \rho)^{1/3} u_2$ , we find that

$$\begin{aligned} \mu_{l,\lambda}(x) &= \int e^{-is_3 \Psi_4(y_2, v, x, \delta, \lambda, l)} a_4(\lambda^{-\frac{1}{3}} y_2, (2^l \rho)^{\frac{1}{3}} v, x, \delta) \\ (13.17) \quad &\quad \times \chi_0((\lambda 2^l \rho)^{-\frac{1}{3}} y_2) \chi_1(s_3) \chi_0(v) dv ds_3 dy_2 \\ &= \int \widehat{\chi_1}(\Psi_4(y_2, v, x, \delta, \lambda, l)) a_4(\lambda^{-\frac{1}{3}} y_2, (2^l \rho)^{\frac{1}{3}} v, x, \delta) \chi_0(v) \chi_0((\lambda 2^l \rho)^{-\frac{1}{3}} y_2) dy_2 dv \end{aligned}$$

with a smooth amplitude  $a_4$  and phase function

$$\begin{aligned} \Psi_4(y_2, v, x, \delta, \lambda, l) &= v \lambda (2^l \rho)^{\frac{1}{3}} (x_1^2 \omega(\delta_1 x_1) - x_2) + \lambda^{\frac{2}{3}} (2^l \rho)^{\frac{1}{3}} \delta_0 v y_2 + \lambda Q_A(x, \delta) \\ &\quad + y_2^3 b(x_1, \lambda^{-\frac{1}{3}} y_2, \delta) + \lambda^{\frac{2}{3}} y_2 Q_D(x, \delta) \end{aligned}$$

We shall write this as

$$(13.18) \quad \Psi_4(y_2, v, x, \delta, \lambda, l) = A + Bv + y_2^3 b(x_1, \lambda^{-\frac{1}{3}} y_2, \delta) + y_2(D + Ev),$$

with

$$\begin{aligned} A &:= \lambda Q_A(x, \delta), & B &:= \lambda 2^{\frac{l}{3}} \rho^{\frac{1}{3}} Q_B(x, \delta), \\ (13.19) \quad D &:= \lambda^{\frac{2}{3}} Q_D(x, \delta), & E &:= \lambda^{\frac{2}{3}} 2^{\frac{l}{3}} (\rho^{\frac{1}{3}} \delta_0). \end{aligned}$$

Here,  $Q_A(x, \delta)$ ,  $Q_B(x, \delta)$  and  $Q_D(x, \delta)$  are as in (12.12).

Applying Lemma 14.2 in the appendix with  $T := (\lambda 2^l \rho)^{1/3} = (\lambda \rho)^{1/3} 2^{l/3} \gg 1$  and  $\delta := \lambda^{-1/3}$  so that  $\delta T = (2^l \rho)^{1/3} \ll 1$ , and subsequently Lemma 12.1 in a similar way as in Subsection 12.2, we find that in analogy with (12.14) we have that

$$(13.20) \quad |\mu_{l,\lambda}(x)| \lesssim (\max\{|A|, |B|, |D|, |E|\})^{-\frac{1}{6}},$$

now with  $A, B, D$  and  $E$  given by (13.19).

In order to estimate  $\nu_{\delta, 1+it}^{IV}(x)$ , we shall again distinguish various cases.

As in the discussion of  $\nu_{\delta, 1+it}^{II}(x)$  in Subsection 12.2, the contributions by those terms in (13.10) for which either  $|D| \gtrsim 1$  and  $|E| \gtrsim 1$ , or  $|A| \gtrsim 1$  and  $|B| \gtrsim 1$ , can easily handled by means of estimate (13.20) (compare with (13.19)).

Consider next the terms for which  $|D| \ll 1$  and  $|E| \ll 1$ . For these terms, it will be useful to re-write  $\mu_{l,\lambda}(x)$  in analogy with (12.15) as

$$(13.21) \quad \mu_{l,\lambda}(x) = \int e^{-is_3(A+Bv)} J(v, s_3) \chi_1(s_3) \chi_0(v) \chi_0(D + Ev) dv ds_3,$$

with

$$(13.22) \quad J(v, s_3) := \int e^{-is_3 \left( y_2^3 b \left( x_1, \lambda^{-\frac{1}{3}} y_2, \delta \right) + y_2(D+Ev) \right)} a_4 \left( \lambda^{-\frac{1}{3}} y_2, (2^l \rho)^{\frac{1}{3}} v, x, \delta \right) \\ \times \chi_0 \left( (\lambda 2^l \rho)^{-\frac{1}{3}} y_2 \right) dy_2.$$

From here we arrive without loss of generality at the following analogue of (12.17):

$$(13.23) \quad \mu_{l,\lambda}(x) = \int g(D + Ev, x, \lambda^{-\frac{1}{3}}, (2^l \rho)^{\frac{1}{3}}, \delta) \widehat{\chi}_1(A + Bv) \chi_0(v) dv,$$

where  $g$  is a smooth function of its bounded arguments.

Recall that we assume that either  $|A| \gtrsim 1$  and  $|B| \ll 1$ , or  $|A| \ll 1$  and  $|B| \gtrsim 1$ , or  $|A| \ll 1$  and  $|B| \ll 1$ .

If  $|A| \gtrsim 1$  and  $|B| \ll 1$ , then we can treat the summation in  $l$  by means of Lemma 5.2, where we choose, for  $\lambda$  fixed,

$$H_{\lambda,x}(u_1, u_2, u_3, u_4) := \int g(D + u_1 v, x, u_3, u_4, \delta) \widehat{\chi}_1(A + u_2 v) \chi_0(v) dv.$$

Then clearly  $\|H_{\lambda,x}\|_{C^1(Q)} \lesssim |A|^{-1}$ , and so after summation in those  $l$  for which  $|E| \ll 1$  and  $|B| \ll 1$ , we can also sum (absolutely) in the  $\lambda$ 's for which  $|A| \gtrsim 1$ . Observe that this requires that  $\gamma(\zeta)$  contains a factor

$$\gamma_1(\zeta) := \frac{2^{\frac{1-\zeta}{2}} - 1}{2^{\frac{1}{3}} - 1}.$$

Consider next the case where  $|B| \gtrsim 1$  and  $|A| \ll 1$ . If we write  $\lambda = 2^j$ , then  $\lambda 2^{l/3} = 2^{k/3}$ , if we put  $k := l + 3j$ . We therefore pass from the summation variables  $j$  and  $l$  to the variables  $j$  and  $k$ , which allows to write  $B = 2^{k/3} \rho^{1/3} Q_B(x)$ . For  $k$  fixed, we then sum first in  $j$  by means of Lemma 5.2, which gives an estimate of order  $O(|B|^{-1})$ , which then in return allows to sum (absolutely) in those  $k$  for which  $|B| \gtrsim 1$ . The application of Lemma 5.2 requires in this case that  $\gamma(\zeta)$  contains a factor

$$\gamma_2(\zeta) := \frac{2^{1-\zeta} - 1}{2^{\frac{2}{3}} - 1}.$$

There remains the sub-case where  $|A| + |B| \ll 1$  and  $|D| + |E| \lesssim 1$ . The summation over all  $l$ 's and  $\lambda$ 's for which these conditions are satisfied can easily be treated by means of the double summation Lemma 8.1 in [21], in a very similar way as in Subsection 12.2.

What remains are the contributions by those  $l$  and  $\lambda$  for which either  $|D| \gtrsim 1$  and  $|E| \ll 1$ , or  $|D| \ll 1$  and  $|E| \gtrsim 1$ .

We begin with the case where  $|E| \gtrsim 1$  and  $|D| \ll 1$ . Then we may assume in addition that  $|B| \ll 1$ , for otherwise by (13.20) we have  $|\mu_{l,\lambda}(x)| \lesssim |E|^{-1/12} |B|^{-1/12}$ , which allows to sum absolutely in  $j$  and  $l$ . In a very similar way, we may also assume that  $|A| \ll 1$ .

Recall next from (13.17) and (13.18) that

$$\begin{aligned} \mu_{l,\lambda}(x) = & \int e^{-is_3(A+y_2^3 b(x_1, \lambda^{-\frac{1}{3}} y_2, \delta) + D y_2 + v(B+E y_2))} a_4(\lambda^{-\frac{1}{3}} y_2, (2^l \rho)^{\frac{1}{3}} v, x, \delta) \\ & \times \chi_1(s_3) \chi_0(v) \chi_0((\lambda 2^l \rho)^{-\frac{1}{3}} y_2) dy_2 dv ds_3. \end{aligned}$$

Again, by our usual argument, we may assume without loss of generality that  $a_4$  is independent of  $v$ . Then we find that, in analogy with (12.18),

$$\begin{aligned} \mu_{l,\lambda}(x) = & \int e^{-is_3(A+y_2^3 b(x_1, \lambda^{-\frac{1}{3}} y_2, \delta))} \widehat{\chi_0}(s_3(B+E y_2)) \\ & \times a_4(\lambda^{-\frac{1}{3}} y_2, (2^l \rho)^{\frac{1}{3}} v, x, \delta) \chi_1(s_3) \chi_0((\lambda 2^l \rho)^{-\frac{1}{3}} y_2) dy_2 ds_3. \end{aligned}$$

We then change the summation variables from  $j, l$  to  $j, k$ , where  $k := 2j + l$ , so that  $E = 2^{k/3} \rho^{1/3} \delta_0$ . Then, for  $k$  fixed, we can treat the summation in  $j$  by means of Lemma 5.2, where we choose

$$\begin{aligned} H_{k,x}(u_1, u_2, u_3, u_4, u_5) := & \int e^{-is_3(u_1+y_2^3 b(x_1, u_2 y_2, \delta))} \widehat{\chi_0}(s_3(u_3+E y_2)) \\ & \times a_4(u_2 y_2, u_4, x, \delta) \chi_1(s_3) \chi_1(v) \chi_0(u_5 y_2) dy_2 ds_3. \end{aligned}$$

Arguing in the same way as in the corresponding case of Subsection 12.2, we find by means of the change of variables  $y_2 \mapsto y_2/E$  that  $\|H_{\lambda,x}\|_{C^1(Q)} \lesssim |E|^{-1}$ , and thus after the summation over the  $\lambda = 2^j$  this allows to subsequently also sum over the  $k$  for which  $|E| \gtrsim 1$ .

There remains the contribution by those  $l$  and  $\lambda$  for which  $|D| \gtrsim 1$  and  $|E| \ll 1$ . Observe that here we have  $|D+E v| \gtrsim 1$  in (13.22).

Applying the change of variables  $y_2 = \lambda^{1/3} t$  in the integral defining  $J(v, s_3)$ , we obtain

$$J(v, s_3) = \lambda^{\frac{1}{3}} \int e^{-is_3 \lambda (t^3 b(x_1, t, \delta) + t(\lambda^{-\frac{2}{3}}(D+E v)))} a_4(t, (2^l \rho)^{\frac{1}{3}} v, x, \delta) \chi_0((2^l \rho)^{-\frac{1}{3}} t) \chi_0(v) \chi_1(s_3) dt.$$

Observe also that by (13.19)

$$(13.24) \quad \lambda^{-\frac{2}{3}} D = Q_D(x, \delta), \quad \lambda^{-\frac{2}{3}} E = \delta_0 \ll 1,$$

so that in particular  $\lambda^{-\frac{2}{3}} |D + vE| \lesssim 1$ .

Arguing in a similar way as in Subsection 12.2, we find that for every  $N \in \mathbb{N}$ ,

$$\begin{aligned}
 J(v, s_3) &= |D + Ev|^{-\frac{1}{4}} a_+ \left( \lambda^{-\frac{1}{3}} |D + Ev|^{\frac{1}{2}}, v, x, (2^l \rho)^{\frac{1}{3}}, \delta \right) \\
 (13.25) \quad &\times \chi_0 \left( (\lambda 2^l \rho)^{-\frac{1}{3}} |D + Ev|^{\frac{1}{2}} \right) e^{-is_3 |D + Ev|^{\frac{3}{2}} q_+ \left( \lambda^{-\frac{1}{3}} |D + Ev|^{\frac{1}{2}}, x, \delta \right)} \\
 &+ |D + Ev|^{-\frac{1}{4}} a_- \left( \lambda^{-\frac{1}{3}} |D + Ev|^{\frac{1}{2}}, v, x, (2^l \rho)^{\frac{1}{3}}, \delta \right) \\
 &\times \chi_0 \left( (\lambda 2^l \rho)^{-\frac{1}{3}} |D + Ev|^{\frac{1}{2}} \right) e^{-is_3 |D + Ev|^{\frac{3}{2}} q_- \left( \lambda^{-\frac{1}{3}} |D + Ev|^{\frac{1}{2}}, x, \delta \right)} \\
 &+ (D + Ev)^{-N} F_N \left( |D + vE|^{\frac{3}{2}}, \lambda^{-\frac{1}{3}} |D + Ev|^{\frac{1}{2}}, v, x, 2^{-\frac{l}{3}}, (2^l \lambda^{-1})^{\frac{1}{3}}, \delta \right),
 \end{aligned}$$

where  $a_{\pm}, q_{\pm}$  and  $F_N$  are smooth functions of their (bounded) variables. Moreover,  $|q_{\pm}(0, x, (2^l \lambda^{-1})^{\frac{1}{3}}, \delta)| \sim 1$ .

We shall concentrate on the first term only. The second term can be treated in the same way as the first one, and the last term can be handled in an even easier way by a similar method, since it is of order  $O(|D|^{-N})$  and, unlike the first term, carries no oscillatory factor.

We denote by

$$\begin{aligned}
 \mu_{l,\lambda}^1(x) &:= \int e^{-is_3 \left[ (A+Bv) + |D+Ev|^{\frac{3}{2}} q_+ \left( \lambda^{-\frac{1}{3}} |D+Ev|^{\frac{1}{2}}, x, \delta \right) \right]} |D + Ev|^{-\frac{1}{4}} \chi_1(s_3) \chi_0(v) \\
 &\times a_+ \left( \lambda^{-\frac{1}{3}} |D + Ev|^{\frac{1}{2}}, v, x, (2^l \rho)^{\frac{1}{3}}, \delta \right) \chi_0 \left( (\lambda 2^l \rho)^{-\frac{1}{3}} |D + Ev|^{\frac{1}{2}} \right) dv ds_3
 \end{aligned}$$

the contribution by the first term in (13.25) to  $\mu_{l,\lambda}(x)$ , and by  $\nu_{\delta,1+it}^1(x)$  the contribution of the  $\mu_{l,\lambda}^1(x)$  to the sum defining  $\nu_{\delta,1+it}^{IV}(x)$ .

Assuming for instance that  $D > 0$ , and making use of (12.22), we here find that the complete phase in the oscillatory integral defining  $\mu_{l,\lambda}^1(x)$  is of the form

$$s_3[A' + B'v + r],$$

with  $A' = \lambda Q_{A'}(x, \delta)$ ,  $B' = \lambda 2^{\frac{l}{3}} Q_{B'}(x, \delta)$ , where

$$\begin{aligned}
 Q_{A'}(x, \delta) &:= Q_A(x, \delta) + Q_D(x, \delta)^{\frac{3}{2}} q_+ \left( Q_D(x, \delta)^{\frac{1}{2}}, x, \delta \right), \\
 Q_{B'}(x, \delta) &:= \rho^{\frac{1}{3}} Q_B(x, \delta) + \frac{1}{2} q'_+ \left( Q_D(x, \delta)^{\frac{1}{2}}, x, \delta \right) \rho^{\frac{1}{3}} Q_D(x, \delta) \delta_0 \\
 &\quad + \frac{3}{2} q'_+ \left( Q_D(x, \delta)^{\frac{1}{2}}, x, \delta \right) \rho^{\frac{1}{3}} Q_D(x, \delta)^{\frac{1}{2}} \delta_0,
 \end{aligned}$$

and where  $r$  is again a bounded error term.

Thus, if  $|B'| \gg 1$ , then an integration by parts in  $v$  shows that

$$|\mu_{l,\lambda}^1(x)| \lesssim |D|^{-\frac{1}{4}} |B'|^{-1}.$$

This estimate allows to control the sum over all  $l$  such that  $|B'| \gg 1$ , and subsequently the sum over all dyadic  $\lambda$  such that  $|D| \gtrsim 1$ , and we arrive at the desired uniform estimate in  $x$  and  $\delta$ .



Next, if  $|B'| \lesssim 1$ , then we can argue in a similar way as before and apply Lemma 5.2 to the summation in  $l$  by putting here

$$\begin{aligned} H_{\lambda,x}(u_1, \dots, u_7) &:= \int e^{-is_3[A' + u_1 v + r(Q_D(x)^{\frac{1}{2}}, D^{-1}u_3, v, x, \delta)]} |D + u_2 v|^{-\frac{1}{4}} \chi_1(s_3) \chi_0(v) \\ &\quad \times a_+(|Q_D(x, \delta) + u_3 v|^{\frac{1}{2}}, v, x, u_4, \delta) \chi_0(|u_6 + u_7 v|^{\frac{1}{2}}) dv ds_3 \end{aligned}$$

and choosing the cuboid  $Q$  in the obvious way. Then we easily see that  $\|H_{\lambda,x}\|_{C^1(Q)} \lesssim |D|^{-1/4}$ , and so after summation in those  $l$  for which  $|B'| \ll 1$ , we can also sum (absolutely) in the  $\lambda$ 's for which  $|D| \gtrsim 1$ . Observe that this requires again that  $\gamma(\zeta)$  contains the factor  $\gamma_1(\zeta)$ .

This concludes the proof of the uniform estimate of  $\nu_{\delta,1+it}^1(x)$  in  $x$  and  $\delta$ , and thus also of estimate (13.9).

The proof of Proposition 11.3 is now complete.

#### 14. APPENDIX: INTEGRAL ESTIMATES OF VAN DER CORPUT TYPE

**Lemma 14.1.** *Let  $b = b(y_1, y_2)$  be a  $C^2$ -function on  $\mathbb{R} \times [-1, 1]$  such that  $b(0, 0) \neq 0$ ,  $\|b(y_1, \cdot)\|_{C^2([-1, 1])} \leq c_1$  for every  $y_1 \in \mathbb{R}$  and*

$$(14.1) \quad |b(y_1, y_2) - b(0, 0)| \leq \varepsilon, \quad \text{and} \quad c_2 |y_2|^{3-j} \leq \left| \partial_{y_2}^j (b(y_1, y_2) y_2^3) \right|, \quad j = 1, 2$$

*for every  $(y_1, y_2) \in \mathbb{R} \times [-1, 1]$ , where  $0 < c_1 \leq c_2$ . Furthermore, let  $Q = Q(y_2)$  be a smooth function on  $[-1, 1]$  such that  $\|Q\|_{C^2([-1, 1])} \leq c_1$ , and let  $A, B, T$  be real numbers so that  $\max\{|A|, |B|\} \geq L$ ,  $T \geq L$  and*

$$(14.2) \quad |A| \leq T^3, \quad |B| \leq T^2,$$

*and let  $r_i = r_i(y_1)$ ,  $i = 1, 2$ , be measurable functions on  $\mathbb{R}$  such that  $|r_i(y_1)| \leq c(1 + |y_1|)$ . For  $\epsilon \geq 0$  and  $N \geq 2$ , we put*

$$\begin{aligned} I_\epsilon(A, B, T) &:= \int_{-T}^T \int_{-\infty}^{\infty} \left( 1 + \left| A - \left( B + Q\left(\frac{y_2}{T}\right) \right) y_2 - b\left(y_1, \frac{y_2}{T}\right) y_2^3 + r_1(y_1) + \frac{y_2}{T} r_2(y_1) \right| \right)^{-N} \\ &\quad \times \left( 1 + |y_1| \right)^{-N} |y_2|^\epsilon dy_1 dy_2. \end{aligned}$$

*Then, for  $N$  sufficiently large, there are constants  $C > 0$  and  $\varepsilon_0 > 0$ , which only depend on the constants  $c, c_1$  and  $c_2$ , such that for all functions  $b$  and  $q$  and all  $A, B, T$  with the assumed properties, and all  $\varepsilon \leq \varepsilon_0$  and  $L \geq C$ , we have*

$$|I_\epsilon(A, B, T)| \leq C \max\{|A|^{\frac{1}{3}}, |B|^{\frac{1}{2}}\}^{\epsilon - \frac{1}{2}}.$$

*Proof.* It will be convenient for the proof to call a constant  $C$  admissible, if it depends only on the constants  $\varepsilon, \epsilon, b(0, 0)$  and  $c, c_1, c_2$  from the statement of the lemma. All constants  $C$  appearing within the proof will be admissible, but may change from line to line.

We begin by the observation that the second assumption in (14.1) implies that also

$$(14.3) \quad c_2 |y_2|^{3-j} \leq \left| \partial_{y_2}^j \left( b(y_1, \tau y_2) y_2^3 \right) \right|, \quad j = 1, 2, \quad |y_2| < \tau^{-1},$$

for every  $\tau > 0$ . Indeed, if we fix  $y_1$  and put  $\psi_\tau(y_2) := b(y_1, \tau y_2) y_2^3$ , then  $\psi_\tau(y_2) = \psi_1(\tau y_2)/\tau^3$ , so that  $\psi_\tau''(y_2) = \psi_1''(\tau y_2)/\tau$ . This immediately leads to (14.3).

Let us first assume that  $|A| \geq L \geq 1$ . Without loss of generality, we may assume that  $A, B \geq 0$  (if necessary, we may change the signs of  $r$  or  $b, q$  as well as of  $y_2$ ). We may then choose  $\alpha, \beta \geq 0$  so that  $A = \alpha^3, B = \beta^2$ .

Next, by convolving  $(1 + |\cdot|)^{-N}$  with a suitable smooth bump function, we may choose a smooth, non-negative function  $\rho$  on  $\mathbb{R}$  which is integrable and such that also its Fourier transform is integrable and so that  $(1 + |x|)^{-N} \leq \rho(x) \leq 2(1 + |x|)^{-N}$ , and put

$$\begin{aligned} J_\epsilon(\alpha, \beta, T) &:= \int_{-T}^T \int_{-\infty}^{\infty} \rho \left( \alpha^3 - (\beta^2 + Q(\frac{y_2}{T})) y_2 - b(y_1, \frac{y_2}{T}) y_2^3 + r_1(y_1) + \frac{y_2}{T} r_2(y_1) \right) \\ &\quad \times \left( 1 + |y_1| \right)^{-N} |y_2|^\epsilon dy_1 dy_2. \end{aligned}$$

It then suffices to prove that

$$(14.4) \quad |J_\epsilon(\alpha, \beta, T)| \leq C \max\{\alpha, \beta\}^{\epsilon - \frac{1}{2}}$$

whenever  $L^{1/3} \leq \alpha \leq T, 0 \leq \beta \leq T$ .

To this end, performing the change of variables  $y_2 = \alpha s$ , we re-write

$$\begin{aligned} J_\epsilon(\alpha, \beta, T) &= \alpha^{1+\epsilon} \int_{-\frac{T}{\alpha}}^{\frac{T}{\alpha}} \int_{-\infty}^{\infty} \rho \left( \alpha^3 \left[ 1 - \left( \gamma + \frac{1}{\alpha^2} Q\left(\frac{\alpha}{T} s\right) \right) s - b(y_1, \frac{\alpha}{T} s) s^3 \right] \right. \\ (14.5) \quad &\quad \left. + r_1(y_1) + s \frac{\alpha}{T} r_2(y_1) \right) \left( 1 + |y_1| \right)^{-N} |s|^\epsilon ds dy_1, \end{aligned}$$

where

$$\gamma := \left( \frac{\beta}{\alpha} \right)^2.$$

**1. Case:**  $\gamma \geq 1$ , i.e.,  $\beta \geq \alpha$ . Changing variables  $s = \gamma^{1/2} t$ , we re-write

$$\begin{aligned} J_\epsilon(\alpha, \beta, T) &= \beta^{1+\epsilon} \int_{-\frac{T}{\beta}}^{\frac{T}{\beta}} \int_{-\infty}^{\infty} \rho \left( \alpha^3 - \beta^3 \left( \left( 1 + \frac{1}{\beta^2} Q\left(\frac{\beta}{T} t\right) \right) t - b(y_1, \frac{\beta}{T} t) t^3 \right) \right. \\ &\quad \left. + r_1(y_1) + t \frac{\beta}{T} r_2(y_1) \right) \left( 1 + |y_1| \right)^{-N} |t|^\epsilon dt dy_1. \end{aligned}$$

Now, there are admissible constants  $C_1, C_2 \geq 1$ , so that if  $|t| \geq C_1$ , then

$$|\alpha^3 - \beta^3 \left( \left(1 + \frac{1}{\beta^2} Q\left(\frac{\beta}{T}t\right)\right)t - b\left(y_1, \frac{\beta}{T}t\right)t^3 \right)| \geq C_2 \beta^3 |t|^3.$$

Notice also that  $|r_1(y_1) + t \frac{\beta}{T} r_2(y_1)| \leq 2c(1 + |y_1|)$ . Integrating separately in  $y_1$  over the sets where  $1 + |y_1| \leq C_2 \beta^3 |t|^3 / (4c)$  and where  $1 + |y_1| > C_2 \beta^3 |t|^3 / (4c)$ , one then easily finds that the contribution  $J_I(\alpha, \beta, T)$  by the region where  $|t| \geq C_1$  to the integral  $J_\epsilon(\alpha, \beta, T)$  can be estimated by

$$|J_I(\alpha, \beta, T)| \leq C \beta^{1+\epsilon} \left( (\beta^3)^{-N} + (\beta^3)^{1-N} \right) \leq 2C \beta^{4+\epsilon-3N}.$$

Assume next that  $|t| < C_1$ . We then choose  $\chi \in C_0^\infty(\mathbb{R})$  such that  $\chi(t) = 1$  when  $|t| \leq C_1$  and  $\chi(t) = 0$  when  $|t| \geq 2C_1$ , and with corresponding control of the derivatives of  $\chi$ . The contribution by the region where  $|t| < C_1$  to the integral  $J_\epsilon(\alpha, \beta, T)$  can then be estimated by

$$J_{II}(\alpha, \beta, T) := \beta^{1+\epsilon} \iint \rho \left( \alpha^3 - \beta^3 \phi_{y_1}(t) + r_1(y_1) \right) \left( 1 + |y_1| \right)^{-N} \chi(t) |t|^\epsilon dt dy_1,$$

where we have set

$$\phi_{y_1}(t) := \left( 1 + \frac{1}{\beta^2} Q\left(\frac{\beta}{T}t\right) - \frac{r_2(y_1)}{\beta^2 T} \right) t - b\left(y_1, \frac{\beta}{T}t\right) t^3, \quad |t| \leq C_1.$$

Recall here that  $T/\beta \geq 1$  (for  $1 \leq |t| \leq 2C_1$ , we may extend the function  $b$  in a suitable way, if necessary).

By Fourier inversion, this can be estimated by

$$\begin{aligned} |J_{II}(\alpha, \beta, T)| &\leq C \beta^{1+\epsilon} \left| \iint \int e^{-i\xi \beta^3 \phi_{y_1}(t)} \chi(t) |t|^\epsilon dt e^{i\xi(\alpha^3 + ir(y_1))} \left( 1 + |y_1| \right)^{-N} \hat{\rho}(\xi) d\xi dy_1 \right| \\ &\leq C \beta^{1+\epsilon} \iint \left| \int e^{-i\xi \beta^3 \phi_{y_1}(t)} \chi(t) |t|^\epsilon dt \right| \left( 1 + |y_1| \right)^{-N} |\hat{\rho}(\xi)| d\xi dy_1. \end{aligned}$$

Now, if  $|r_2(y_1)/(\beta^2 T)| \geq \frac{1}{2}$ , then  $c(1 + |y_1|) \geq \beta^2 T/2 \geq 1$ , so trivially the integration in  $y_1$  yields that

$$|J_{II}(\alpha, \beta, T)| \leq C \beta^{1+\epsilon} (\beta^2 T)^{1-N} \leq C \beta^{3+\epsilon-2N}$$

for every  $N \in \mathbb{N}$ .

So, assume that  $|r_2(y_1)/(\beta^2 T)| < \frac{1}{2}$ . Then the phase  $\phi_{y_1}(t)$  has no degenerate critical point on the support of  $\chi(t)$ , if we assume  $\epsilon$  to be sufficiently small, then  $b$  is a small perturbation of the constant function  $b(0, 0)$ , in the sense of (14.1). It is then easily verified that our assumptions (and in particular (14.3)) imply that  $\phi_{y_1}(t)$  satisfies the hypotheses of the van der Corput type Lemma 2.2 in [21], with  $M = 2$  and constants  $c_1, c_2 > 0$  which are admissible, provided we choose  $L$  sufficiently large. Therefore, the lemma shows that the inner integral with respect to  $t$  is bounded by  $C(1 + |\xi| \beta^3)^{-1/2}$ , which implies that

$$|J_{II}(\alpha, \beta, T)| \leq C \beta^{1+\epsilon} (\beta^3)^{-\frac{1}{2}} = C \beta^{\epsilon-\frac{1}{2}}.$$

**2. Case:**  $\gamma < 1$ , i.e.,  $\beta < \alpha$ . Then there are admissible constants  $C_3, C_4 \geq 1$  so that if  $|s| \geq C_3$ , then  $\alpha^3 |1 - (\gamma + \frac{1}{\alpha^2} Q(\frac{\alpha}{T}s)) s - b(y_1, \frac{\alpha}{T}s) s^3| \geq C_4 \alpha^3 |s|^3$  in (14.5). Arguing in a similar way as in the first case, this implies that the contribution  $J_{III}(\alpha, \beta, T)$  by the region where  $|s| \geq C_3$  to the integral  $J_\epsilon(\alpha, \beta, T)$  in (14.5) can be estimated by

$$|J_{III}(\alpha, \beta, T)| \leq C \alpha^{4+\epsilon-3N}.$$

Similarly, there are admissible constants  $C_5, C_6 > 0$  so that if  $|s| \leq C_5$ , then  $|1 - (\gamma + \frac{1}{\alpha^2} Q(\frac{\alpha}{T}s)) s - b(y_1, \frac{\alpha}{T}s) s^3| \geq C_6$  in (14.5), and this implies that the contribution  $J_{IV}(\alpha, \beta, T)$  by the region where  $|s| \leq C_5$  to the integral  $J_\epsilon(\alpha, \beta, T)$  can be estimated by

$$|J_{IV}(\alpha, \beta, T)| \leq C \alpha^{4+\epsilon-3N}.$$

Finally, on the set where  $C_5 < |s| < C_3$ , the phase  $\phi_{y_1}(s) := (\frac{r_2(y_1)}{\alpha^2 T} - (\gamma + \frac{1}{\alpha^2} Q(\frac{\alpha}{T}s)))s - b(y_1, \frac{\alpha}{T}s) s^3$  has again no degenerate critical point, and we conclude in a similar way as in the first case that the contribution  $J_V(\alpha, \beta, T)$  by this region can be estimated by

$$|J_V(\alpha, \beta, T)| \leq C \alpha^{1+\epsilon} (\alpha^3)^{-\frac{1}{2}} = C \alpha^{\epsilon-\frac{1}{2}}.$$

Combining all these estimates, we arrive at the conclusion of the lemma when  $|A| \geq 1$ .

Finally, when  $|A| \leq L$  and  $|B| \geq L$ , then we may indeed assume without loss of generality that  $\alpha = 0$ . Arguments very similar to those that we have applied before then show that  $|J_\epsilon(\alpha, \beta, T)| \leq C \beta^{\epsilon-\frac{1}{2}}$ , which concludes the proof of the lemma. Q.E.D.

**Lemma 14.2.** *Let  $b = b(y)$  be a  $C^2$ -function on  $[-1, 1]$  such that  $b(0) \neq 0$  and  $\|b\|_{C^2([-1, 1])} \leq c_1$ . Furthermore, let  $A$  and  $B$  be real numbers, and let  $\delta_0 \in ]0, 1[$ . For  $T \geq L$  and  $\delta > 0$  such that  $\delta < 1$  and  $\delta T \leq \delta_0$ , we put*

$$I(A, B) := \int_{\mathbb{R}} |\rho(A + By + b(\delta y)y^3)| \chi_0\left(\frac{y}{T}\right) dy,$$

where  $\rho \in \mathcal{S}(\mathbb{R})$  denotes a fixed Schwartz function and  $\chi_0 \in C_0^\infty(\mathbb{R})$  a non-negative bump function supported in the interval  $[-1, 1]$ . Then, for  $\delta_0$  sufficiently small and  $L$  sufficiently large, we have

$$(14.6) \quad |I(A, B)| \leq C \left(1 + \max\{|A|^{\frac{1}{3}}, |B|^{\frac{1}{2}}\}\right)^{-\frac{1}{2}},$$

where the constant  $C$  depends only on  $c_1, \delta_0, \rho$  and  $\chi_0$ , but not on  $A, B, T$  and  $\delta$ .

*Proof.* In a similar way as in the preceding proof, we may dominate the function  $|\rho|$  by a non-negative Schwartz function. Let us therefore assume without loss of generality that  $\rho \geq 0$ .

Let us first assume that  $\max\{|A|, |B|\} \geq L$ . If  $|A| \leq T^3/\delta_0^3$  and  $|B| \leq T^2/\delta_0^2$ , estimate (14.6) follows easily from the previous Lemma 14.1 (just replace  $\chi_0(\cdot/T)$  by the characteristic function of the interval  $[-1/\delta, 1/\delta]$ ).

Next, if  $|B| > T^2/\delta_0^2$ , using Fourier inversion, we may estimate

$$(14.7) \quad |I(A, B)| \leq C \int |\int e^{is\phi(y)} \chi_0(\frac{y}{T}) dy| |\hat{\rho}(s)| ds,$$

where  $\phi(y) := By + b(\delta y)y^3$ . And, by means of integration by parts, we obtain that

$$|\int e^{is\phi(y)} \chi_0(\frac{y}{T}) dy| \leq C|s|^{-1} \int_{-T}^T \left( \left| \frac{\phi''(y)}{\phi'(y)^2} \right| + \left| \frac{1}{T\phi'(y)} \right| \right) dy.$$

Since for  $|y| \leq T$  we have

$$|\phi'(y)| = |B - y^2(3b(\delta y) + \frac{\delta y}{3}b'(\delta y))| \geq \frac{|B|}{2} \quad \text{and} \quad |\phi''(y)| \leq CT,$$

if we choose  $\delta_0$  sufficiently small, we see that  $|\int e^{is\phi(y)} \chi_0(\frac{y}{T}) dy| \leq C/|sB|$  for  $\delta_0$  sufficiently small. On the other hand, trivially we have  $|\int e^{is\phi(y)} \chi_0(\frac{y}{T}) dy| \leq CT$ , and taking the geometric mean of these estimates and using that  $T < \delta_0|B|^{1/2}$  we find that

$$|\int e^{is\phi(y)} \chi_0(\frac{y}{T}) dy| \leq C|s|^{-\frac{1}{2}}|B|^{-\frac{1}{4}}.$$

In combination with (14.7) this confirms estimate (14.6) also in this case.

There remains the case where  $|A| > T^3/\delta_0^3$  and  $|B| \leq T^2/\delta_0^2$ . Here we may estimate

$$|A + By + b(\delta y)y^3| \geq |A| - |B|T - \|b\|_\infty T^3 \geq \frac{|A|}{2},$$

provided  $\delta_0$  is sufficiently small. This implies that  $|I(A, B)| \leq C|A|^{-N}T \leq C|A|^{-N+1/3}$  for every  $N \in \mathbb{N}$ , which is stronger than what we need for (14.6).

Finally, if  $\max\{|A|, |B|\} < L$ , then the rapid decay of  $\rho$  easily implies that  $|I(A, B)| \leq C$ . Q.E.D.

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DEPARTMENT OF MATHEMATICS, SAMARKAND STATE UNIVERSITY, UNIVERSITY BOULEVARD  
15, 140104, SAMARKAND, UZBEKISTAN

*E-mail address:* ikromov1@rambler.ru

MATHEMATISCHES SEMINAR, C.A.-UNIVERSITÄT KIEL, LUDEWIG-MEYN-STRASSE 4, D-24098  
KIEL, GERMANY

*E-mail address:* mueller@math.uni-kiel.de

*URL:* <http://analysis.math.uni-kiel.de/mueller/>